QCD phase diagram at $\mu \neq 0$

1. Standard lore:

   One transition, chiral = deconfined, “semicircle”

2. Large number of colors, N_c:

   Two transitions, chiral ≠ deconfinement

   “Quarkyonic” matter

   Confined, chirally symmetric baryons: massive, parity doubled.

3. QCD?

   Perhaps: phase intermediate between nuclear matter and “just” quarks

McLerran & RDP, 0706.2191. Hidaka, McLerran, & RDP 0803.0279

The first semicircle

Cabibbo and Parisi '75: Exponential (Hagedorn) spectrum limiting temperature, *or* transition to new, "unconfined" phase. One transition.

Punchline today: below for chiral transition, deconfinement splits off at finite μ.

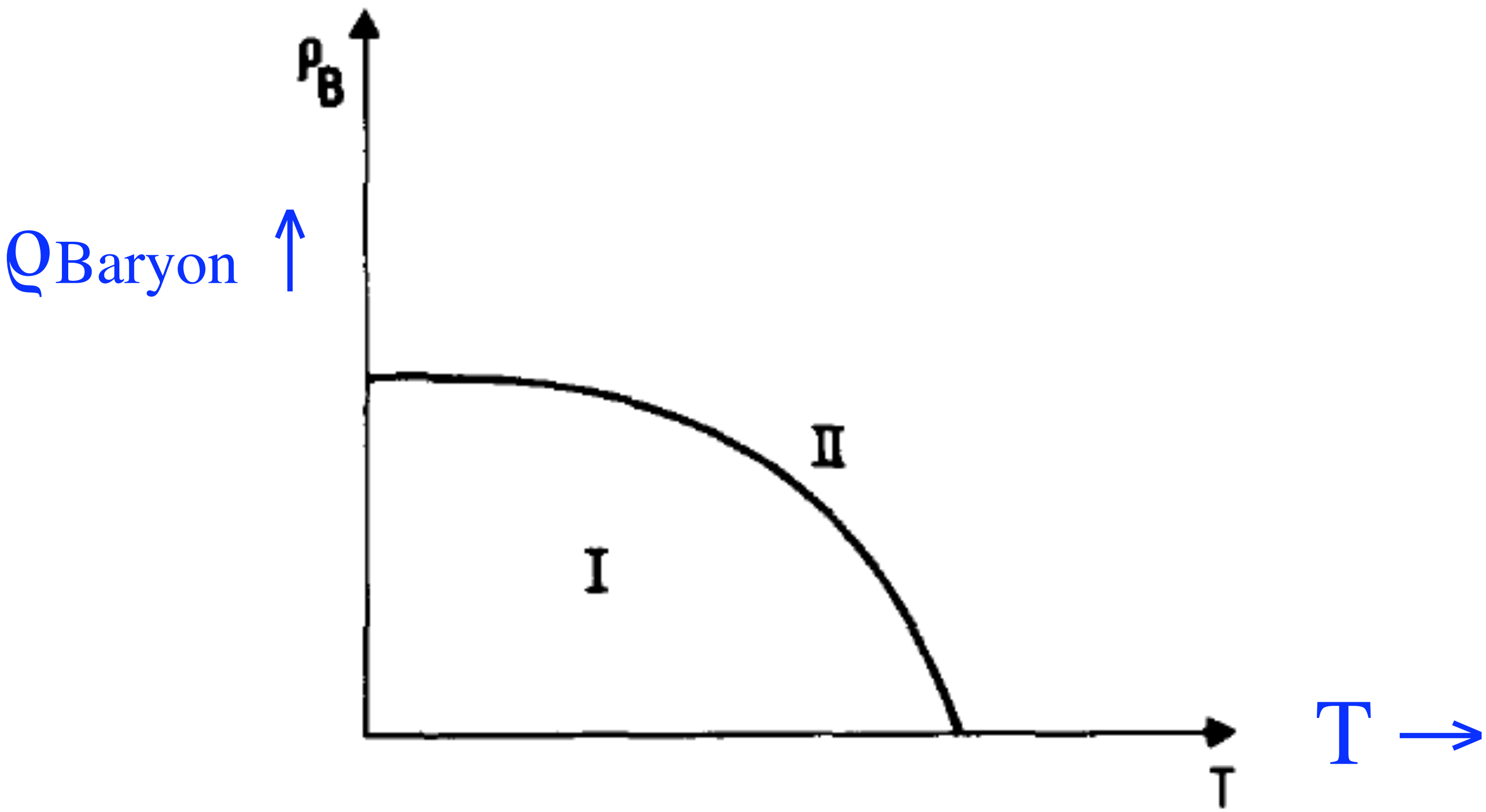


Fig. 1. Schematic phase diagram of hadronic matter. ρ_B is the density of baryonic number. Quarks are confined in phase I and unconfined in phase II.

Two transitions?

Shuryak '82, RDP '82: Natural to have two transitions:

T_d = deconfinement, T_χ = restoration of chiral symmetry

$T_\chi > T_d$: for $T_\chi > T > T_d$, *deconfined but massive quarks*. Very natural.

Chiral symmetry breaking is due to a (sufficiently) strong coupling.

At T_d , coupling is large enough to deconfine, but not restore chiral sym.

$T_d > T_\chi$: for $T_d > T > T_\chi$, *confined but chirally symmetric quarks (?)*.

Casher '79: confining theories break chiral symmetry. Meson in rest frame:

Blue: direction of motion. Red: spin.

A quark going rightward mixes with a quark going left.

For massless quarks, spin along (or opposite) to direction of motion: helicity.

Thus a confined, *massless* quark *must* flip its helicity.

But in QCD, gluon interactions conserve helicity.

Helicity is flipped by mass, so ok if chiral symmetry is broken.

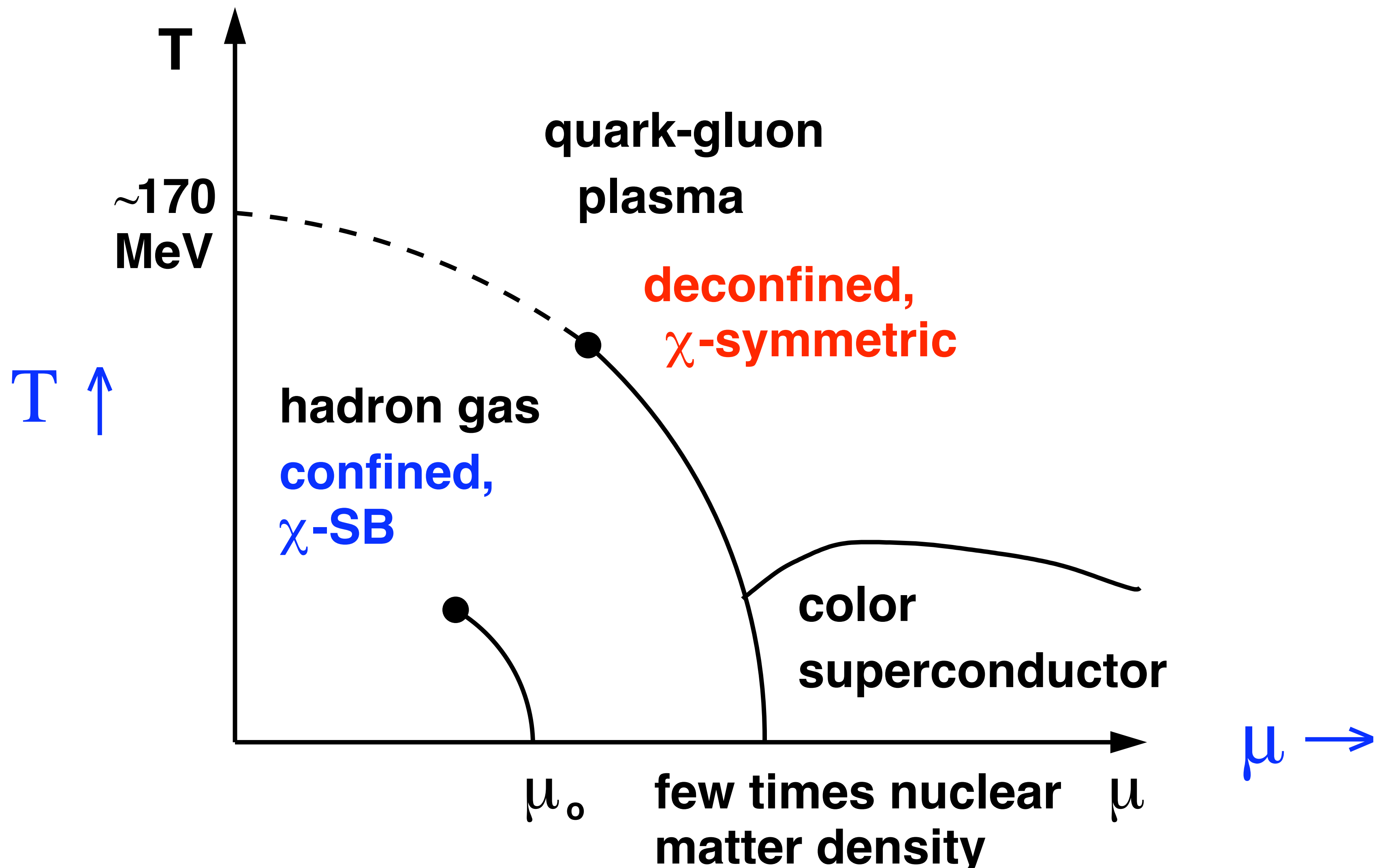
Phase diagram, ~ '06

Lattice, $T \neq 0$, $\mu = 0$: two possible transitions; *one* crossover, *same* T.

Karsch hep-lat/0601013. Remains crossover for $\mu \neq 0$?

Stephanov, Rajagopal, & Shuryak hep-ph/9806219, /9903292, /0010100

Critical end point where crossover turns into first order transition



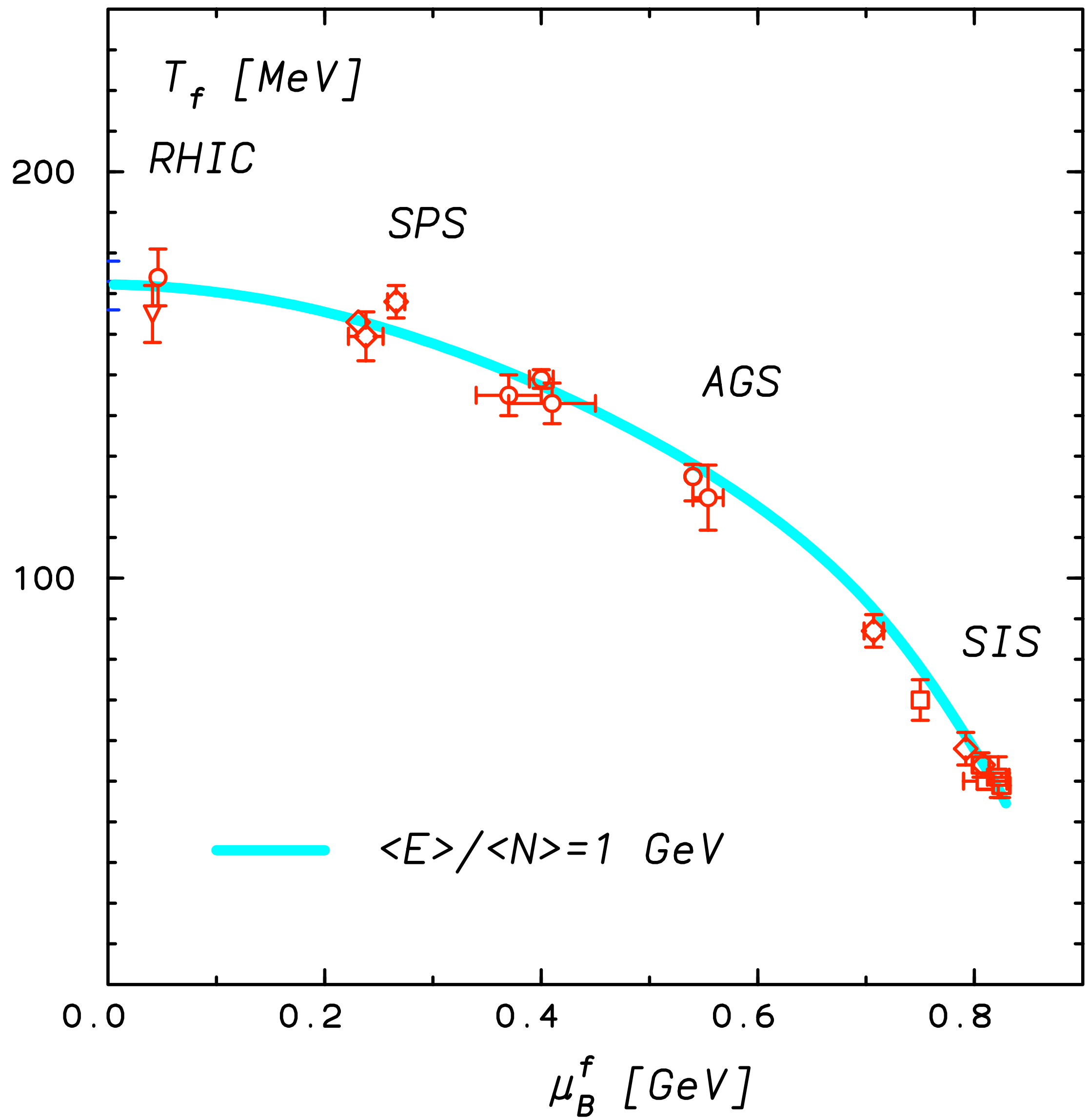
Experiment: freezeout line

Cleymans & Redlich nucl-th/9906065...Kraus, Cleymans, Oeschler, Redlich 0808.0611

Line for chemical equilibriation at freezeout ~ semicircle.

N.B.: for T = 0, goes down to ~ nucleon mass.

T ↑



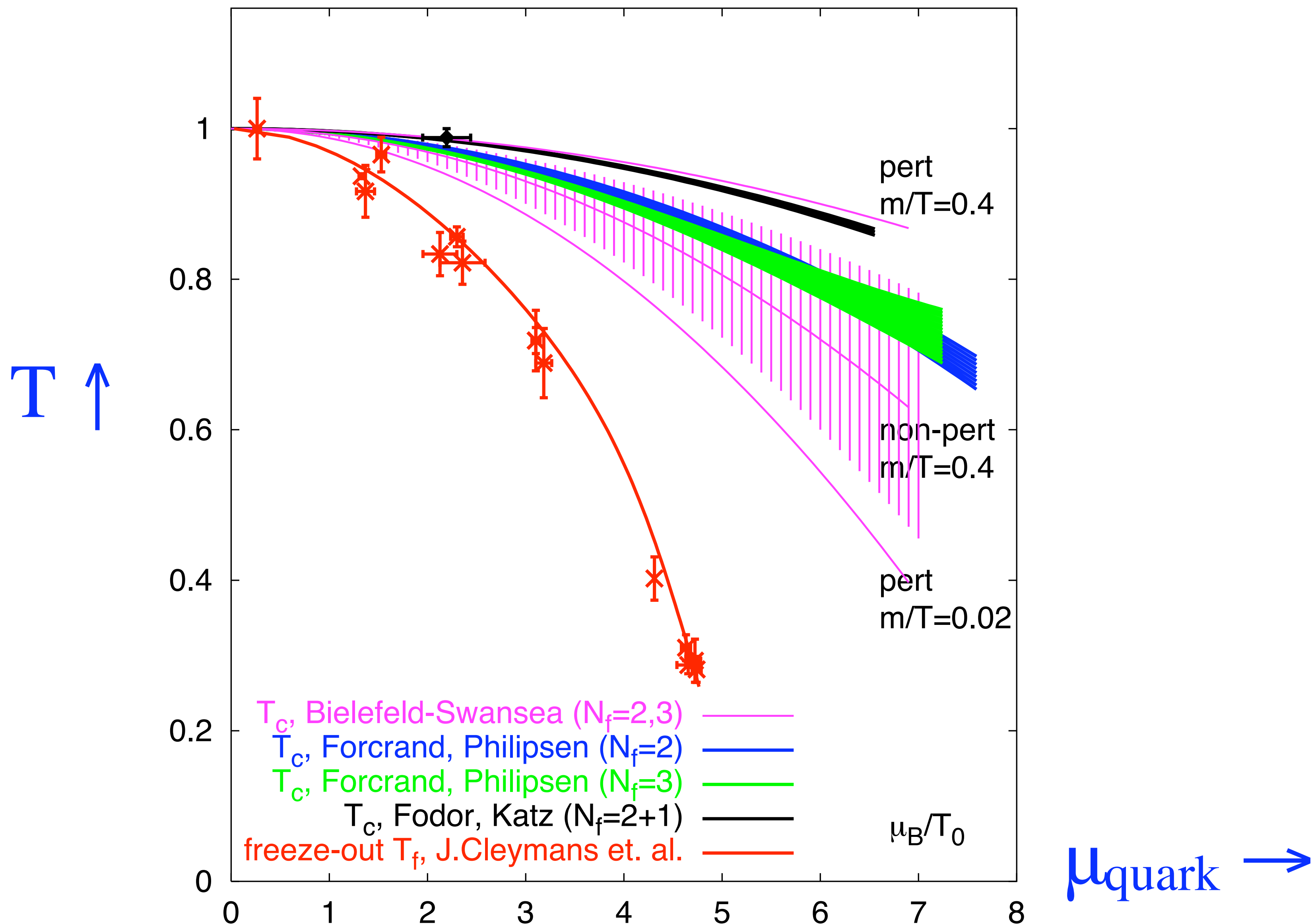
μ_{Baryon} →

Experiment vs. Lattice

Lattice “transition” appears *above* freezeout line?

Fodor, Katz, & Schmidt hep-lat/0701022; Schmidt ‘07

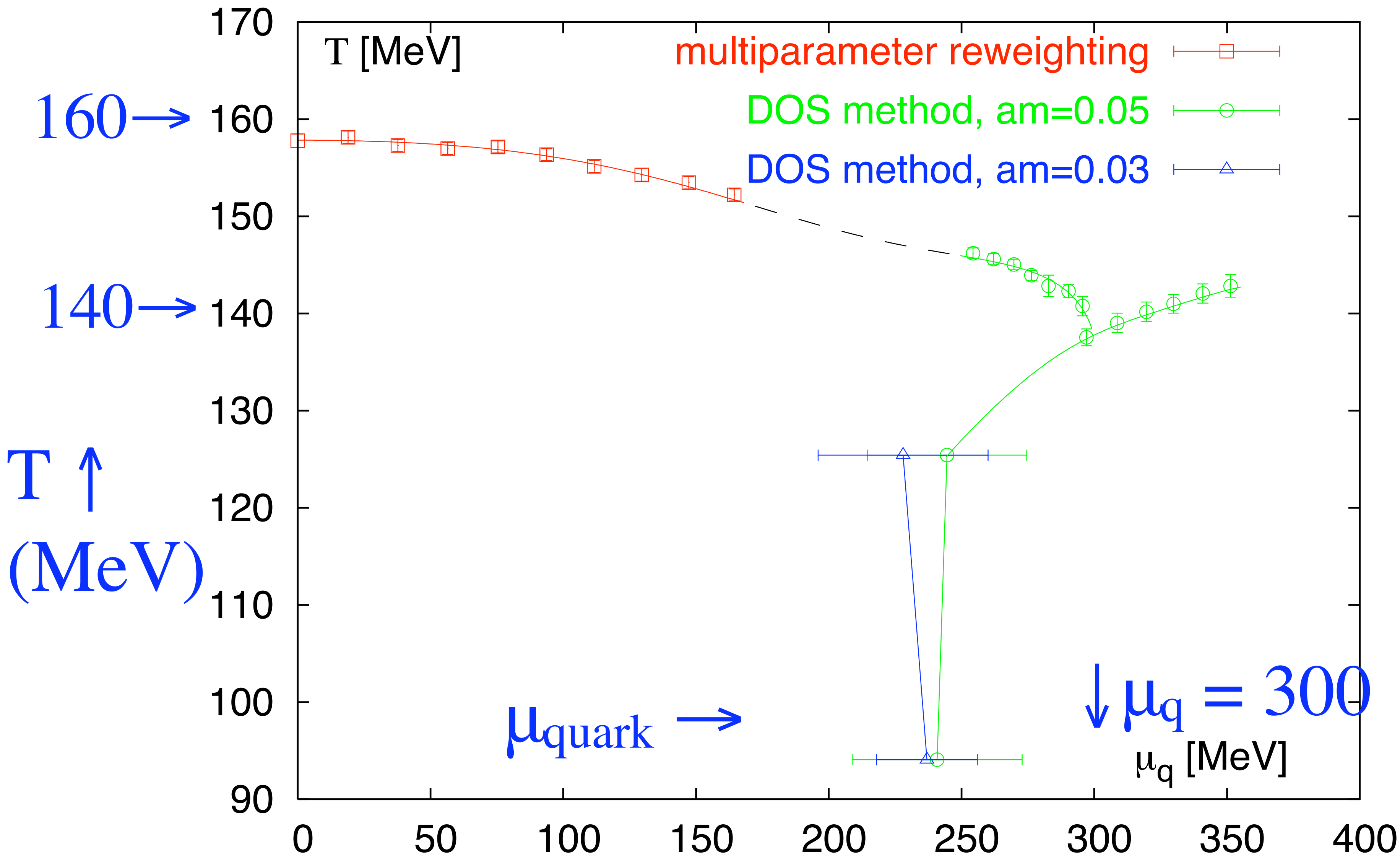
N.B.: *small* change in T_c with μ?



Lattice T_c , vs μ

Rather small change in T_c vs μ? Depends where μ_c is at $T = 0$.

Fodor, Katz, & Schmidt hep-lat/0701022



EoS of nuclear matter

Akmal, Panharipande, & Ravenhall nucl-th/9804027:

Equation of State for nuclear matter, T=0

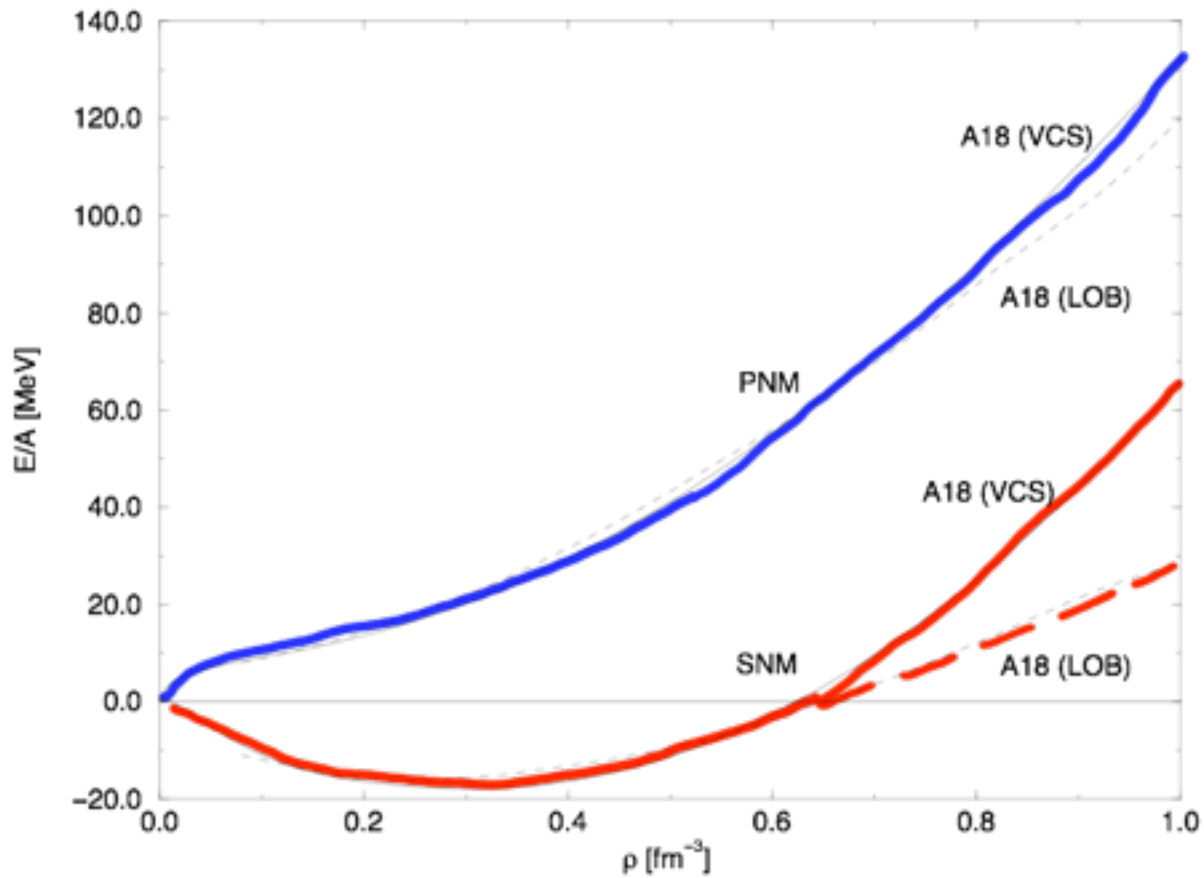
E/A = energy/nucleon. Fits to various nuclear potentials: A18= Argonne 18…

PNM = pure neutron matter. SNM = symmetric nuclear matter (equal #'s n's, p's)

Binding energy of nuclear matter ~ 15 MeV!

Much smaller than any natural hadronic scale: f_π , Λ_{MS}…

E/A ↑



ρ_{Baryon} →

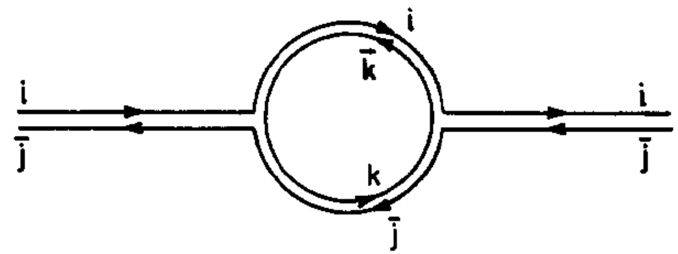
Expansion in large N_c

't Hooft '74: let $N_c \to \infty$, with $\lambda = g^2 N_c$ fixed.

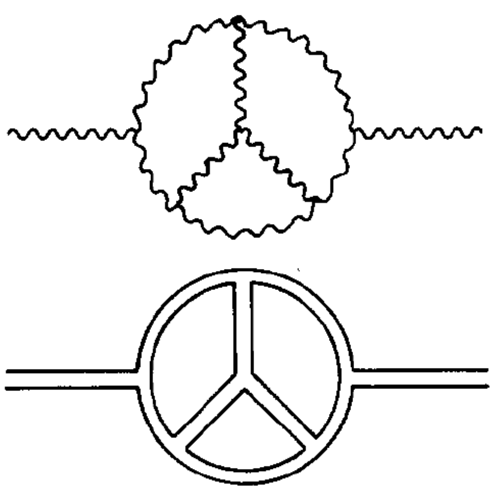
$\sim N_c^2$ gluons in adjoint representation, vs $\sim N_c$ quarks in fundamental rep. $\Rightarrow$

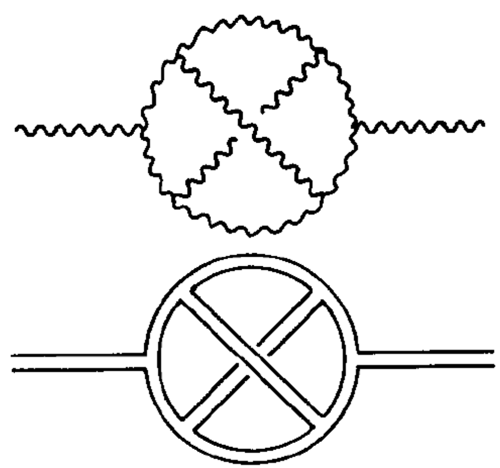
large N_c dominated by *gluons* (iff N_f = # quark flavors *small*)

Double line (birdtrack) notation:



$\sim g^2 N_c = \lambda$



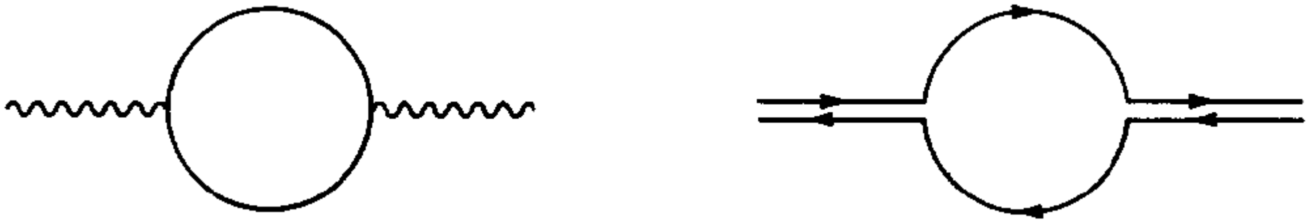


Planar diagram, $\sim \lambda^2$

Non-planar diagram, $\sim \lambda^2 / N_c$

Suppressed by $1/N_c$. Trace terms also $1/N_c$

Quark loops suppressed at large N_c



$$\sim g^2 = \lambda \times \frac{1}{N_c}$$

Quark loops are suppressed at large N_c *if* N_f , # quark flavors, is held fixed

Thus: limit of: large N_c , *small* N_f

Quarks introduced as external sources.

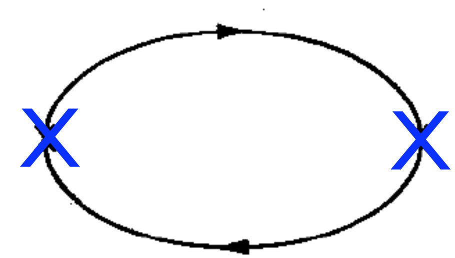
Analogous to “quenched” approximation, expansion about $N_f = 0$.

Veneziano ‘78: take both N_c and N_f large.
Can use baryon number as order parameter: Hidaka, McLerran, & RDP.

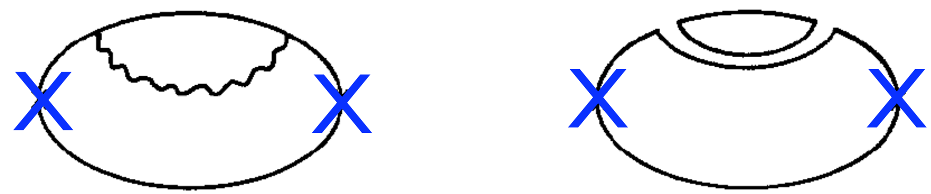
Form factors at large N_c

J ~ (gauge invariant) mesonic current

$$< J(x)J(0) > \sim N_c$$



Infinite # of planar diagrams for $< J\, J >$:



Confinement => sum over mesons, form factors ~ $N_c^{1/2}$

$$< J(x)J(0) > \sim \int d^4p\, e^{ip\cdot x} \sum_n < 0|J|n > \frac{1}{p^2+m_n^2} < n|J|0 >$$

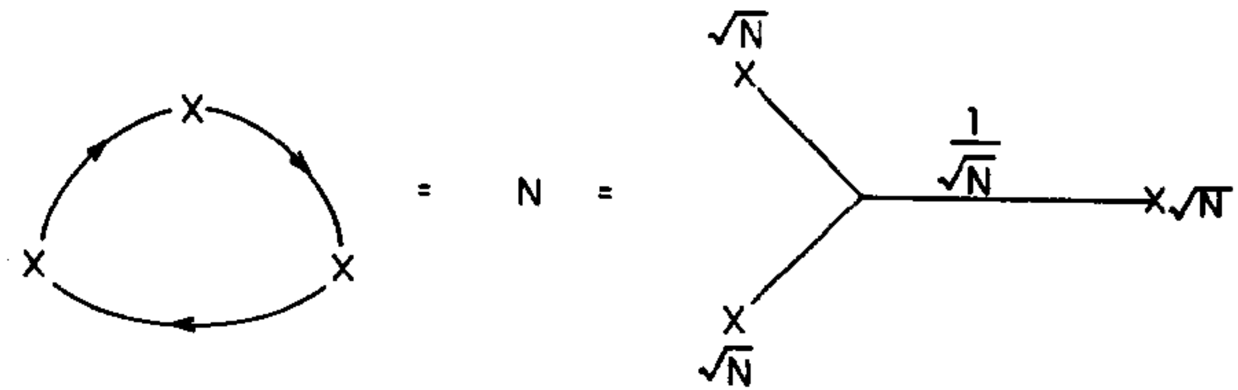
$$< J(x)J(0) > \sim \; N_c \quad \Rightarrow \; < 0|J|n > \sim \sqrt{N_c} \;\text{ if }\; m_n \sim 1$$

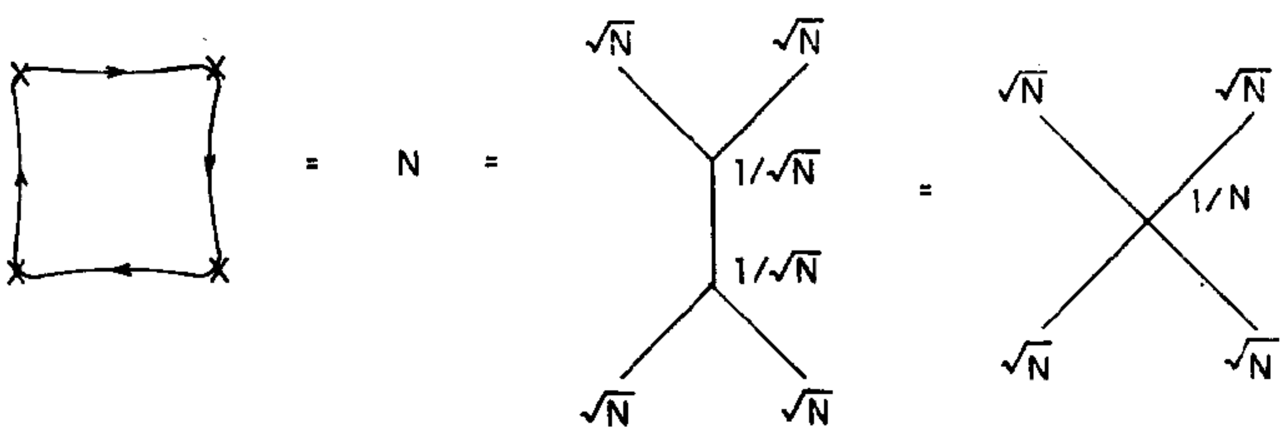
Mesons & glueballs *free* at $N_c = \infty$

With form factors $\sim N_c^{1/2}$, 3-meson couplings $\sim 1/N_c^{1/2}$; 4-meson, $\sim 1/N_c$
For glueballs, 3-glueball couplings $\sim 1/N_c$, 4-glueball $\sim 1/N_c^2$

Mesons and glueballs don't interact at $N_c = \infty$.

Large N limit *always* (some) classical mechanics Yaffe '82



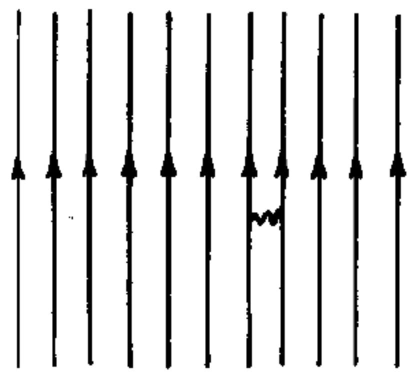


Baryons at large N_c

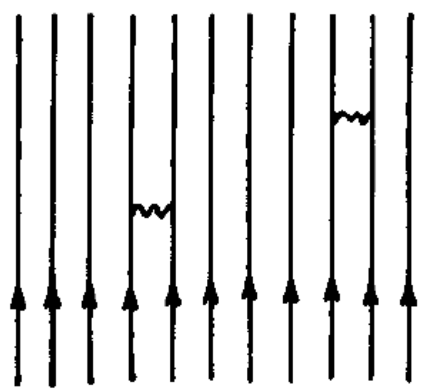
Witten '79: Baryons have N_c quarks, so nucleon mass $M_N \sim N_c \Lambda_{QCD}$.

Baryons like "solitons" of large N_c limit (~ Skyrmion)

Leading correction to baryon mass:



$$g^2 \times N_c \times N_c \sim \lambda \, N_c$$

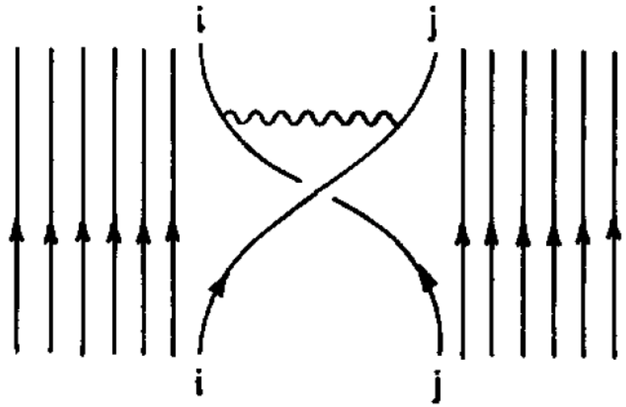


Appears $\sim g^4 N_c^4 \sim \lambda^2 N_c^2$?

No, iteration of average potential, mass still $\sim N_c$.

Baryons are *not* free at $N_c = \infty$

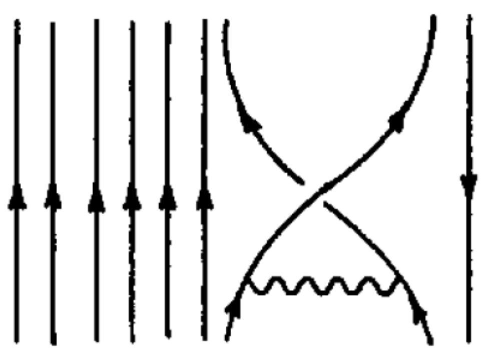
Baryons interact strongly. Two baryon scattering ~ N_c :



$$g^2 \times N_c \times N_c \sim \lambda\, N_c$$

Scattering of three, four... baryons also ~ N_c

Mesons also interact strongly with baryons, ~ N_c^0 ~ 1



$$g^2 \times N_c \sim \lambda$$

Towards the phase diagram at $N_c = \infty$

As example, consider gluon polarization tensor at zero momentum.

(at leading order, ~ Debye mass^2 , gauge invariant)

$$\Pi^{\mu\mu}(0) = g^2 \left(\left(N_c + \frac{N_f}{2} \right) \frac{T^2}{3} + \frac{N_f \mu^2}{2\pi^2} \right) = \lambda \, \frac{T^2}{3} \ , \ N_c = \infty$$

For $\mu \sim N_c^0 \sim 1$, at $N_c = \infty$ the gluons are blind to quarks.

When $\mu \sim 1$, deconfining transition temperature $T_d(\mu) = T_d(0)$

Chemical potential only matters when larger than mass:

$\mu_{Baryon} > M_{Baryon}$. Define $m_{quark} = M_{Baryon}/N_c$; so $\mu > m_{quark}$.

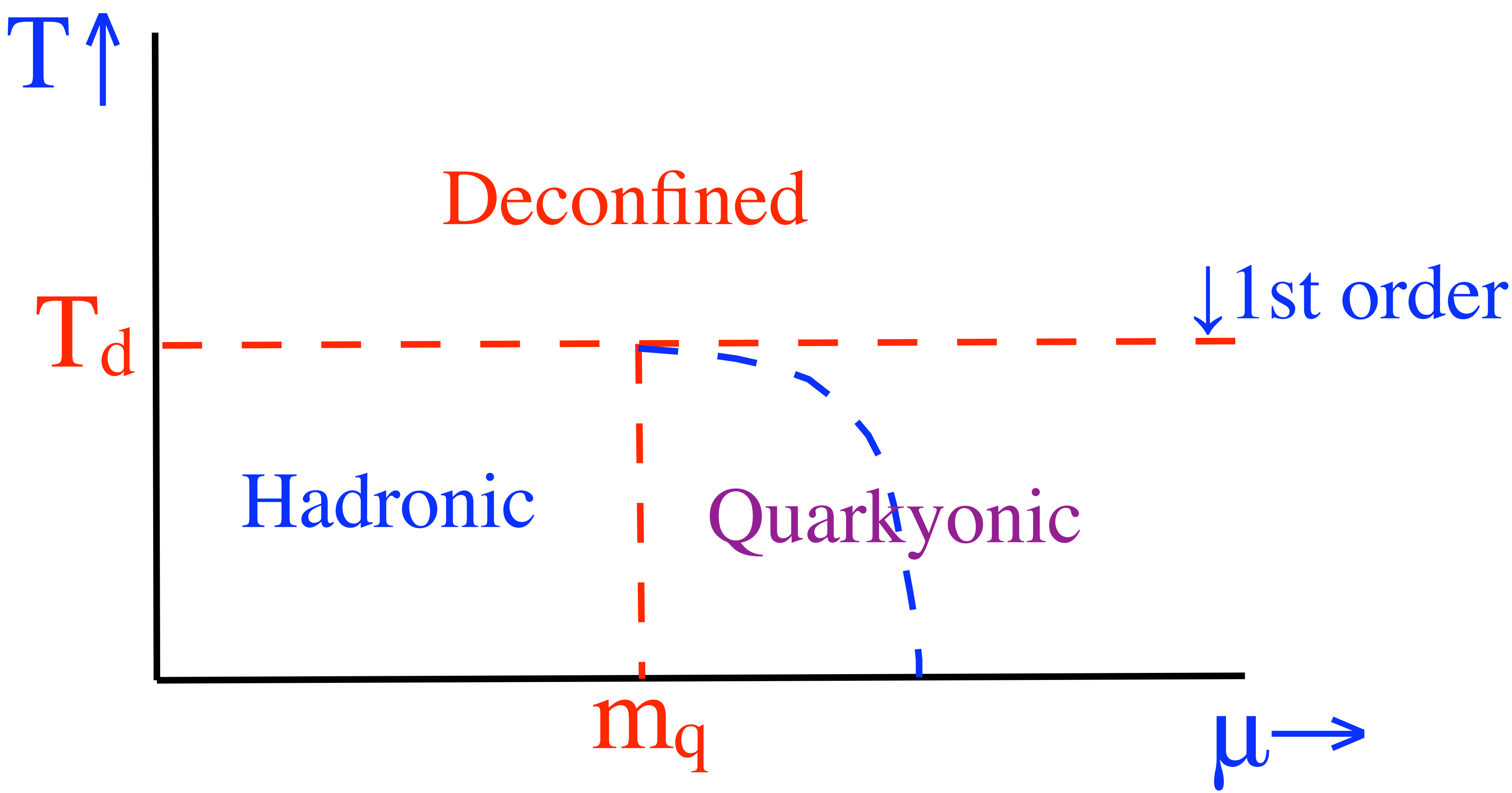
"Box" for $T < T_c$; $\mu < m_{quark}$: confined phase baryon free, since their mass ~ N_c

Thermal excitation ~ $\exp(-m_B/T) \sim \exp(-N_c) = 0$ at large N_c.

So hadronic phase in "box" = mesons & glueballs only, *no* baryons.

Phase diagram at $N_c = \infty$, I

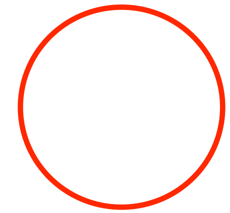
At *least* three phases. At large N_c, can use pressure, P, as order parameter.
Hadronic (confined): $P \sim 1$. Deconfined, $P \sim N_c^2$. Thorn '81; RDP '84...
$P \sim N_c$: quarks or baryonic = "quark-yonic". Chiral symmetry restoration?
L. McLerran & RDP, 0706.2191
N.B.: mass threshold at m_q neglects (possible) nuclear binding, Son.



Nuclear matter at large N_c

$\mu_{Baryon} = \sqrt{k_F^2 + M^2}$, k_F = Fermi momentum of baryons.
Pressure of ideal baryons density times energy of non-relativistic baryons:

$$P_{\text{ideal baryons}} \sim n(k_F)\,\frac{k_F^2}{M} \sim \frac{1}{N_c}\;\frac{k_F^5}{\Lambda_{QCD}}$$

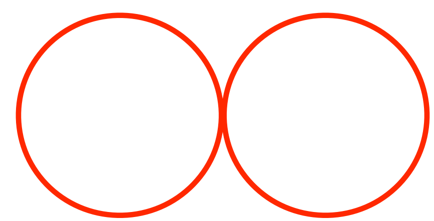


This is small, ~ $1/N_c$. The pressure of the I = J tower of resonances is as small:

$$\delta P_{\text{resonances}} \sim \frac{1}{M}\,\frac{k_F^8}{\Lambda_{QCD}^3} \sim \frac{1}{N_c}\;\frac{k_F^8}{\Lambda_{QCD}^4}$$

Two body interactions are huge, ~ N_c in pressure.

$$\delta P_{\text{two body int.'s}} \sim N_c\,\frac{n(k_F)^2}{\Lambda_{QCD}^2} \sim N_c\;\frac{k_F^6}{\Lambda_{QCD}^2}$$



At large N_c , nuclear matter is dominated by potential, not kinetic terms!
Two body, three body... interactions *all* contribute ~ N_c .

Window of nuclear matter

Balancing $P_{\text{ideal baryons}} \sim P_{\text{two body int.'s}}$, interactions important very quickly,

$$k_F \sim \frac{1}{N_c^2}\,\Lambda_{QCD}$$

For such momenta, only two body interactions contribute.

By the time $k_F \sim 1$, *all* interactions terms contribute $\sim N_c$ to the pressure.

But this is *very* close to the mass threshold,

$$\mu - m_q = \frac{\mu_B - M}{N_c} = \frac{k_F^2}{2MN_c} \sim \frac{1}{N_c^2}\,k_F^2$$

Hence "ordinary" nuclear matter is only in a *very* narrow window.

One quickly goes to a phase with pressure $P \sim N_c$.

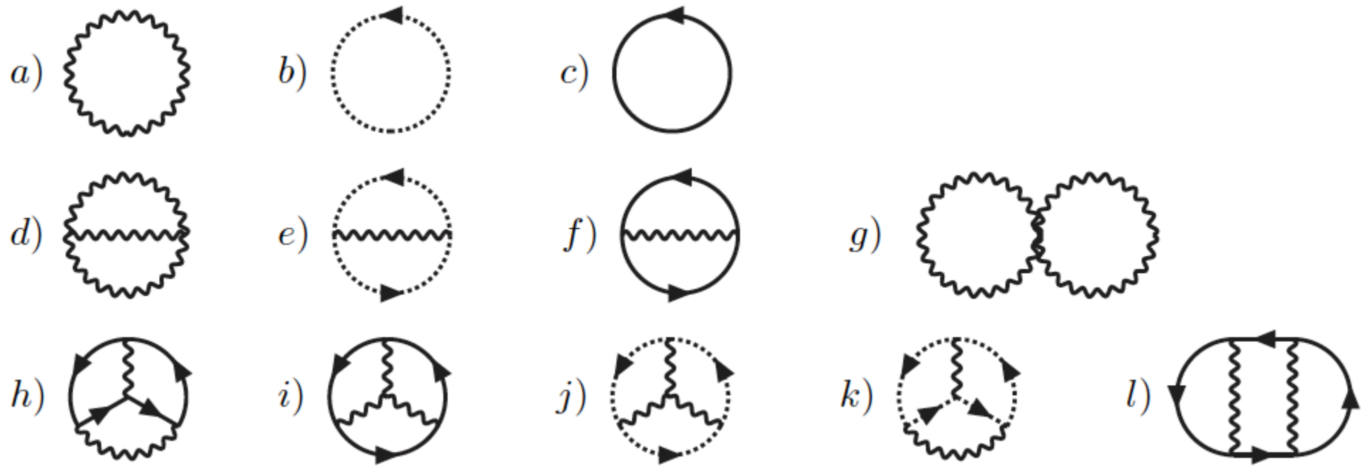
So are they baryons, or quarks?

Perturbative pressure

At high density, $\mu >> \Lambda_{QCD}$, compute $P(\mu)$ in QCD perturbation theory.

To $\sim g^4$, (Freedman & McLerran)^4 '77

Ipp, Kajantie, Rebhan, & Vuorinen, hep-ph/0604060



At $\mu \neq 0$, only diagrams with at least one quark loop contribute. Still...

$$P_{\mathrm{pert.}}(\mu) \sim N_c N_f \; \mu^4 \; F_0(g^2(\mu/\Lambda_{QCD}), N_f)$$

For $\mu >> \Lambda_{QCD}$, but $\mu \sim N_c^0 \sim 1$, calculation reliable.

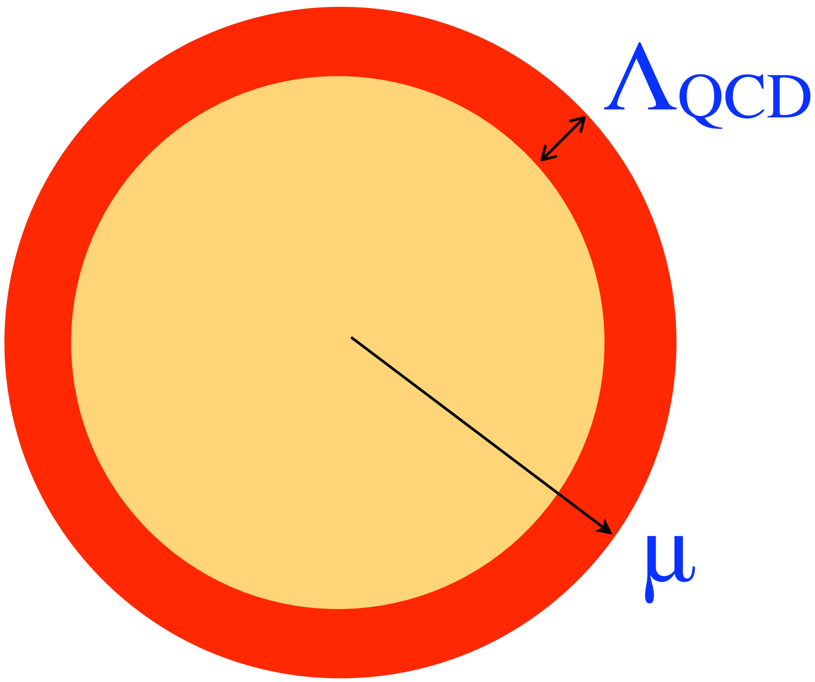
Compute $P(\mu)$ to $\sim g^6$? No "magnetic mass" at $\mu \neq 0$, well defined $\forall$ $(g^2)^n$.

“Quarkyonic” phase at large N_c

As gluons blind to quarks at large N_c, for $\mu \sim N_c^0 \sim 1$, *confined* phase for $T < T_d$

This includes $\mu >> \Lambda_{QCD}$! Central puzzle. We suggest:

To the right: Fermi sea =>



Deep in the Fermi sea, $k << \mu$,
looks like quarks.

But: within $\sim \Lambda_{QCD}$ of the Fermi surface,
confinement => *baryons*

We term combination “*quark-yonic*”

OK for $\mu >> \Lambda_{QCD}$. When $\mu \sim \Lambda_{QCD}$, baryonic “skin” entire Fermi sea.

But what about chiral symmetry breaking?

Skyrmions and $N_c = \infty$ baryons

Witten '83; Adkins, Nappi, Witten '83: Skyrme model for baryons

$$\mathcal{L} = f_\pi^2 \, \mathrm{tr}|V_\mu|^2 + \kappa \, \mathrm{tr}[V_\mu, V_\nu]^2 \; , \; V_\mu = U^\dagger \partial_\mu U \; , \; U = e^{i\pi/f_\pi}$$

Baryon soliton of pion Lagrangian: $f_\pi \sim N_c^{1/2}$, $\kappa \sim N_c$, mass $\sim f_\pi^2 \sim \kappa \sim N_c$.

Above Lagrangian simplest form: presumably (?) *infinite* series in V_μ.

Single baryon: at $r = \infty$, $\pi^a = 0$, $U = 1$. At $r = 0$, $\pi^a = \pi \, r^a/r$.
Baryon number topological: Wess & Zumino '71; Witten '83.

Huge degeneracy of baryons: multiplets of isospin and spin, $I = J$: $1/2 \ldots N_c/2$.
Obvious as collective coordinates of soliton, coupling spin & isospin

Dashen & Manohar '93, Dashen, Jenkins, & Manohar '94:
Baryon-meson coupling $\sim N_c^{1/2}$,
Cancellations from extended $SU(2 \, N_f)$ symmetry.

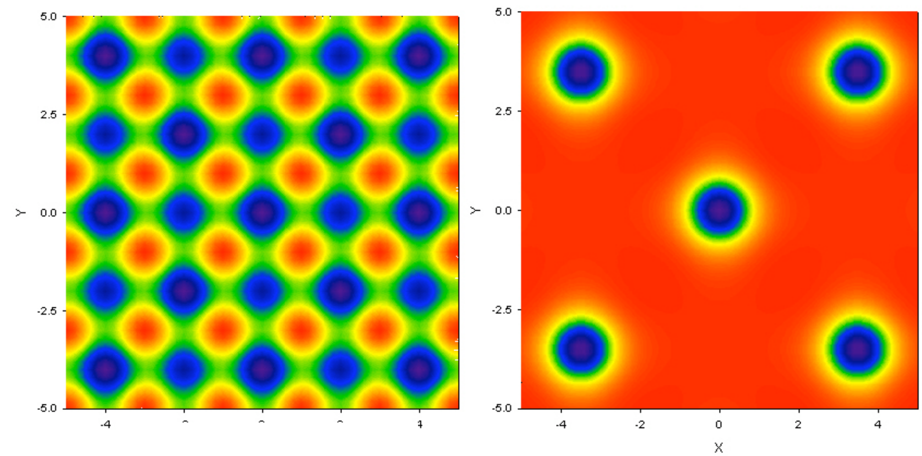
Skyrmion crystals

Skyrmion crystal: soliton periodic in space.

Kutschera, Pethick & Ravenhall (KPR) '84; Klebanov '85 + ...

Lee, Park, Min, Rho & Vento, hep-ph/0302019

Park, Lee, & Vento, 0811.3731:



At low density, chiral symmetry broken by Skyrme crystal, as in vacuum.

Chiral symmetry *restored* at nonzero density: $< U > = 0$ in each cell.

Goldhaber & Manton '87: due to "half" Skyrmion symmetry in each cell.

Forkel, Jackson et al, '89: excitations *are* chirally symmetric.

Easiest to understand with "spherical" crystal, KPR '84, Manton '87.

Take same boundary conditions as a single baryon, but for sphere of radius R:

At $r = 0$: $\pi^a = 0$. At $r = R$, $\pi^a = \pi\, r^a/r$. Density one baryon/$(4\,\pi\, R^3/3)$.

At high density, term $\sim \kappa$ dominates, so energy density $\sim$ baryon density$^{4/3}$.

Like perturbative QCD! Accident of simplest Skyrme Lagrangian.

Casher's argument in a Fermi sea

What about Casher's argument?

In vacuum, can neglect scattering with baryons, as baryon (anti-baryon) pairs are exponentially suppressed, exp(-N).

In a Fermi sea, though, baryon's are there by construction. Thus a massless quark can scatter off of a baryon in the Fermi sea:

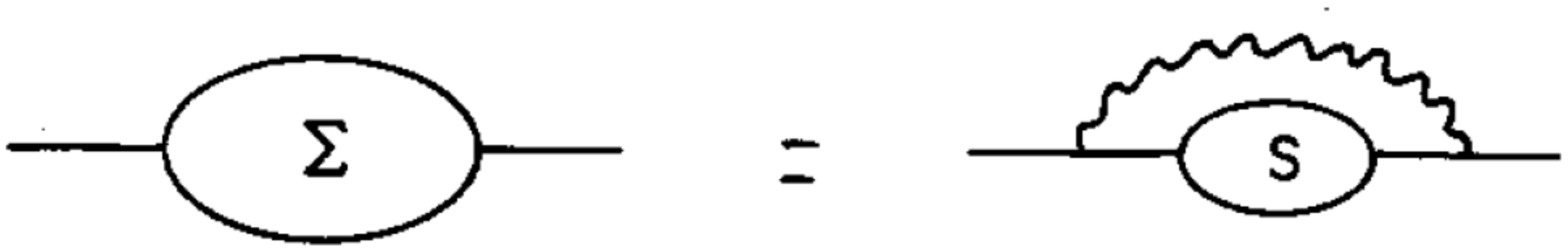
Quark baryon scattering is large. Thus a quark can scatter off of a baryon, in the Fermi sea. This process can flip the helicity, since quark baryon scattering does not conserve helicity.

Argument special to having a Fermi sea: one needs baryons about.

Standard Casher argument *does* apply at $T \neq 0$, $\mu = 0$, without a Fermi sea.

Schwinger-Dyson equations at large N_c: 1+1 dim.'s

't Hooft '74: as gluons blind to quarks at large Nc, S-D eqs. simple for quark:
Gluon propagator, and gluon quark anti-quark vertex unchanged.
To leading order in $1/N_c$, only quark propagator changes:



't Hooft '74: in 1+1 dimensions, single gluon exchange generates linear potential,

$$g_{2D}^2 \int dk \, \frac{e^{ikr}}{k^2} \sim g_{2D}^2 \, r$$

In vacuum, Regge trajectories of confined mesons. Baryons?

Solution at $\mu \neq 0$? Should be possible, not yet solved.

M. Thies, hep-th/0601049, C. Boehmer, U. Fritsch, S. Kraus, & M. Thies, 0807.2571
Gross-Neveu model has crystalline structure at $\mu \neq 0$

Quarkyonic matter via Schwinger-Dyson

Glozman & Wagenbrunn 0709.3080, 0805.4799; Glozman 0812.1101:

In 3+1 dimensions, confining gluon propagator, $1/(k^2)^2$ as $k^2 \to 0$:

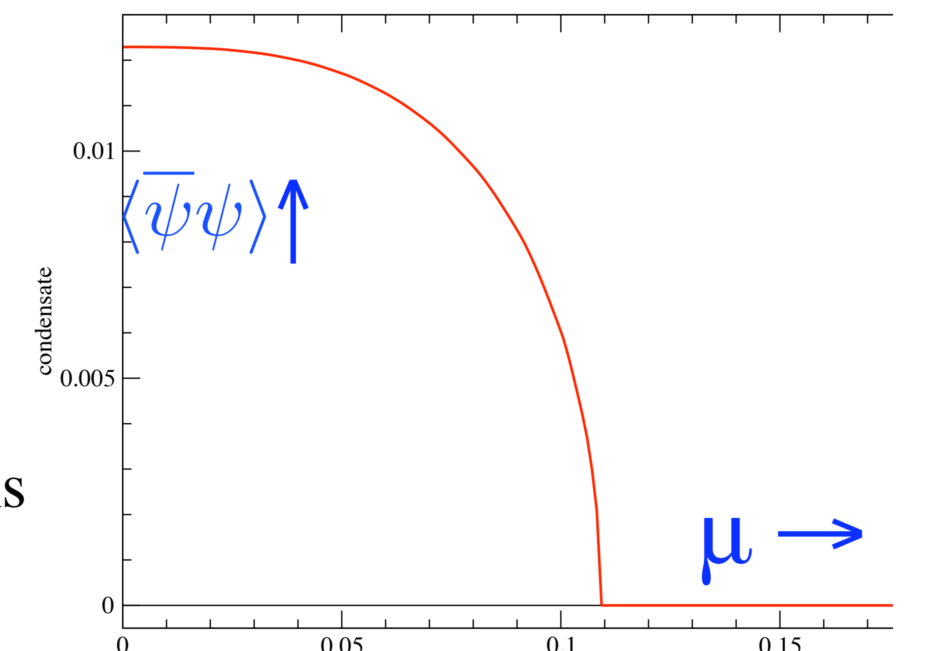
$$g^2 \int d^3k \, \frac{e^{ikr}}{k^2} \left(1 + \frac{\sigma}{k^2}\right) \sim g^2 \, \sigma \, r \; , \; r \to \infty$$

σ = string tension. Very similar to 1+1 dimensions. $\mu = 0$: $\langle \overline{\psi} \psi \rangle = (.23 \sqrt{\sigma})^3$

Take Schwinger-Dyon eq. at large N_c: confinement unchanged by $\mu \neq 0$.
Treat μ by usual cutoff in momentum space: for confining system, same as $\mu \neq 0$?

Chiral symmetry restoration: $\mu_\chi = .11 \sqrt{\sigma}$
Transition *second* order: not evident.
Also: all infrared divergences *cancel*.

No nuclear matter:
restore chiral symmetry before Fermi sea forms



Asymptotically large μ, grows with N_c

For $\mu \sim (N_c)^p$, $p > 0$, gluons feel the effect of quarks. Perturbatively,

$$P_{\mathrm{pert.}}(\mu, T) \sim N_c N_f \, \mu^4 \, F_0 \; , \; N_c N_f \, \mu^2 \, T^2 \, F_1 \; , \; N_c^2 \, T^4 \, F_2 \; .$$

First two terms from quarks & gluons, last only from gluons. Two regimes:

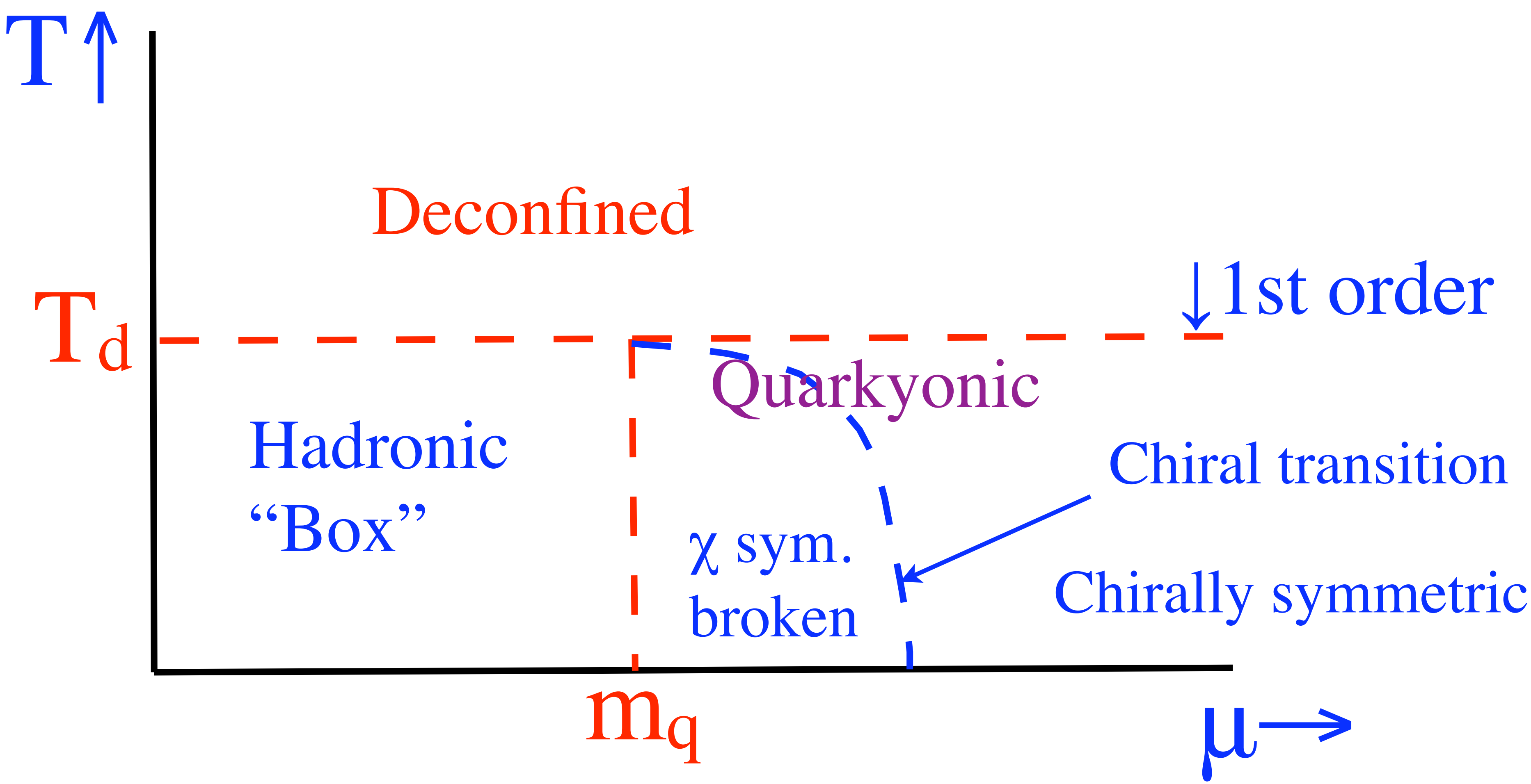
$\mu \sim N_c^{1/4} \Lambda_{QCD}$: $N_c \mu^4 F_0 \sim N_c^2 F_2 \sim N_c^2 >> N_c \mu^2 F_1 \sim N_c^{3/2}$.
Gluons & quarks contribute equally to pressure; quark cont. T-independent.

$\mu \sim N_c^{1/2} \Lambda_{QCD}$: New regime: $m^2_{Debye} \sim g^2 \mu^2 \sim 1$, so gluons feel quarks.

$N_c \mu^4 F_0 \sim N_c^3 >> N_c \mu^2 F_1$, $N_c^2 F_2 \sim N_c^2$.
Quarks dominate pressure, T-independent.

Eventually, first order deconfining transition can either:
end in a critical point, or bend over to $T = 0$: ?

Phase diagram at $N_c = \infty$, II



We suggest: quarkyonic phase includes chiral trans. Order by usual arguments.

Mocsy, Sannino & Tuominen hep-th/0308135:

splitting of transitions in effective models

But: quarkyonic phase *confined*. Chirally symmetric baryons?

Chirally symmetric baryons: parity doubling

B. Lee, '72; DeTar & Kunihiro '89; Jido, Oka & Hosaka, hep-ph/0110005; Zschiesche et al nucl-th/0608044. Wigner-Weyl mode: *two* baryon multiplets. One usual nucleon, + parity. Other state with - parity. *Perhaps* N*(1535)

Transform *opposite* under chiral transformations:

$$\psi_{L,R} \rightarrow U_{L,R}\, \psi_{L,R} \;\; ; \;\; \chi_{L,R} \rightarrow U_{R,L}\, \chi_{L,R}$$

With two multiplets, can form chirally symmetric (parity even) mass term:

$$m_0 \left(\psi_L \chi_R - \psi_R \chi_L + \chi_R \psi_L - \chi_L \psi_R \right)$$

Also: usual sigma field, $\Phi \rightarrow U_L \, \Phi \, U_R^\dagger$, couplings for linear sigma model:

$$g_1 \, \psi_L \, \Phi \, \psi_R + g_2 \, \chi_R \, \Phi \, \chi_L$$

In component form:

$$g_1 \, \psi (\sigma + i\gamma_5 \overline{\tau} \cdot \overline{\pi}) \psi + g_2 \, \overline{\chi} (\sigma - i\gamma_5 \overline{\tau} \cdot \overline{\pi}) \chi$$

N.B.: difference in signs for axial coupling between two fields.

Parity doubled baryons

Letting $<\sigma> = \sigma_0$, obtain two states with split masses

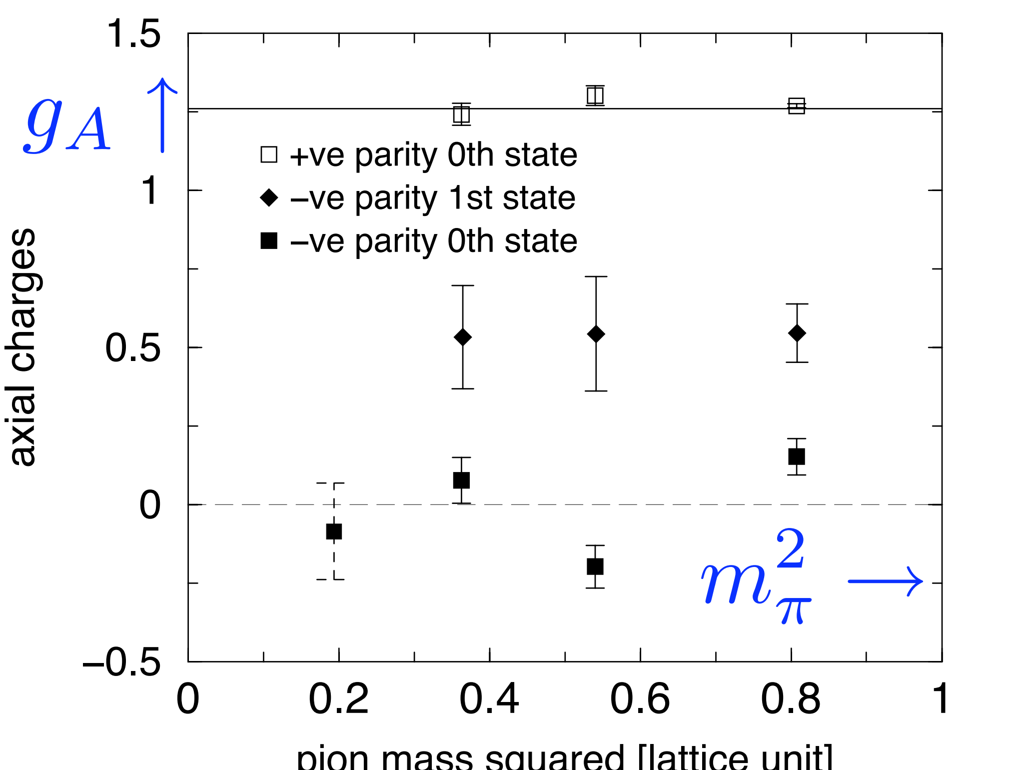
$$m_{N\pm} = \frac{1}{2}\left(\sqrt{(g_1+g_2)^2\sigma_0^2 + 4m_0^2} \mp (g_1-g_2)\sigma_0\right)$$

Looks ok: when $\sigma_0 = 0$, states degenerate, with mass m_0.
Can arrange couplings so that $g_1 \sim 13$, $g_2 \sim 3$ are natural; m_0 small, ~ 300 MeV.

But: as parity partner, g_A should be +1 for N, -1 for parity partner.
Takahashi & Kunihiro 0801.4707:
lattice, two flavors dynamical quarks:

$g_A \sim 0$ for parity partner!
How do the masses of nucleon... change,
mix as density increases?

If quarkyonic right, *must* be
parity doubled model*: what is it?*
Glozman hep-ph/0701081



Anomalies in vacuum

Usually, relate anomalies to pions. Adler '69: can compute $\pi^0 \to \gamma\gamma$.

Let V = vector current, couples to photons. Conserved.

A = axial vector current, couples to pions. Isospin-3 comp. = axial anomaly:

$$\partial_\mu V^\mu = 0 \quad ; \quad \partial_\mu A^\mu = -\frac{e^2 N_c}{48\pi^2} F_{\mu\nu}\widetilde{F}^{\mu\nu}$$

Consider "VVA":

$$\Gamma^{\mu\nu\sigma}(P_1, P_2) = \langle V^\mu(P_1) V^\nu(P_2) A^\sigma(Q)\rangle \; ; \; -Q = P_1 + P_2$$

This satisfies

$$P_1^\mu \, \Gamma^{\mu\nu\sigma} = P_2^\nu \, \Gamma^{\mu\nu\sigma} = 0 \qquad\qquad Q^\sigma \, \Gamma^{\mu\nu\sigma} = -\frac{e^2 N_c}{12\pi^2} \, \epsilon^{\mu\nu\sigma\kappa} \, P_1^\sigma P_2^\kappa$$

Construct

$$\mathcal{T}^{\mu\nu} = e^2 Q^2 \langle V^\mu(P_1) V^\nu(P_2)\pi\rangle \qquad\qquad \epsilon_1^\mu \epsilon_2^\nu \, \mathcal{T}^{\mu\nu} = g_{\pi\gamma\gamma} \, \epsilon^{\mu\nu\sigma\kappa} \epsilon_1^\mu \epsilon_2^\nu P_1^\sigma P_2^\kappa$$

and take out the one pion pole term, using PCAC:

$$\langle 0|A^\sigma|\pi\rangle = iQ^\sigma f_\pi \qquad\qquad \widetilde{\Gamma}^{\mu\nu\sigma} = \Gamma^{\mu\nu\sigma} + f_\pi \frac{Q^\sigma}{Q^2} \mathcal{T}^{\mu\nu}$$

Anomalies in vacuum, cont.'d

Most general amplitude which is Bose symmetric under photon exchange, $P_1\ \mu \leftrightarrow P_2\ \nu$. Also Lorentz invariant.

$$\widetilde{\Gamma}^{\mu\nu\sigma} = T_1 \epsilon^{\mu\nu\sigma\kappa}(P_1 - P_2)^\kappa + T_2(\epsilon^{\mu\sigma\kappa\lambda} P_2^\nu - \epsilon^{\nu\sigma\kappa\lambda} P_1^\mu) P_1^\kappa P_2^\lambda$$
$$+ T_3(\epsilon^{\mu\sigma\kappa\lambda} P_1^\nu - \epsilon^{\nu\sigma\kappa\lambda} P_2^\mu) P_1^\kappa P_2^\lambda$$

Current conservation: $P_1^\mu\ \widetilde{\Gamma}^{\mu\nu\sigma} = P_2^\nu\ \widetilde{\Gamma}^{\mu\nu\sigma} = 0$

$$T_1 + P_1^2\ T_2 + P_1 \cdot P_2\ T_2 = 0$$

Axial anomaly:

$$Q^\sigma\ \widetilde{\Gamma}^{\mu\nu\sigma} = f_\pi \mathcal{T}^{\mu\nu} - \frac{e^2 N_c}{12\pi^2}\ \epsilon^{\mu\nu\sigma\kappa}\ P_1^\sigma P_2^\kappa$$

$$-2T_1 = f_\pi g_{\pi\gamma\gamma} - \frac{e^2 N_c}{12\pi^2}$$

Now consider on shell photons and pions: $P_1^2 = P_2^2 = Q^2 = 0$.

Then $T_1 = 0$, which relates pion-photon coupling as:

$$f_\pi\ g_{\pi\gamma\gamma} = \frac{e^2 N_c}{12\pi^2}$$

$T_1 = 0$ is Sutherland-Veltman theorem.

Anomalies in medium

Itoyama & Mueller'83; RDP, Trueman & Tytgat hep-ph/9702362:

Now do the same in a medium. Just have extra vector for rest frame, n^μ

$$\widetilde{\Gamma}^{\mu\nu\sigma} = T_1 \epsilon^{\mu\nu\sigma\kappa}(P_1 - P_2)^\kappa + T_2(\epsilon^{\mu\sigma\kappa\lambda} P_2^\nu - \epsilon^{\nu\sigma\kappa\lambda} P_1^\mu) P_1^\kappa P_2^\lambda$$
$$+ T_3(\epsilon^{\mu\sigma\kappa\lambda} P_1^\nu - \epsilon^{\nu\sigma\kappa\lambda} P_2^\mu) P_1^\kappa P_2^\lambda$$
$$+ T_4\, n\cdot Q\, \epsilon^{\mu\nu\kappa\lambda} P_1^\kappa P_2^\lambda n^\sigma + T_5(n\cdot P_2\, \epsilon^{\mu\sigma\lambda\kappa} n^\nu - n\cdot P_1\, \epsilon^{\nu\sigma\lambda\kappa} n^\mu) P_1^\lambda P_2^\kappa$$

Current conservation:

$$T_1 + P_1^2\, T_2 + P_1\cdot P_2\, T_3 + (n\cdot P_1)^2\, T_5 = 0$$

Axial anomaly:

$$-2T_1 + (n\cdot Q)^2 T_4 = f_\pi g_{\pi\gamma\gamma} - \frac{e^2 N_c}{12\pi^2}$$

On the mass shell:

$$f_\pi\, g_{\pi\gamma\gamma} = \frac{e^2 N_c}{12\ \ {}^2} + (n\cdot Q)^2 T_4 + 2(n\cdot P_1)^2 T_5$$

Here f_π and $g_{\pi\gamma\gamma}$ are functions of temperature, density....

The axial anomaly does relate these to amplitudes, but they don't vanish, even on mass shell. Sutherland-Veltman *fails* in a medium.

Verified by explicit calculation to one loop order: at $T \neq 0$, $T_4 = 0$, $T_5 \neq 0$

‘t Hooft anomaly condition

‘t Hooft ‘79; Coleman & Witten ‘80

Consider a current (M = flavor matrix) $$J^\mu = \overline{\psi}\, M(1+\gamma_5)\gamma^\mu\, \psi$$

Construct the three point function, $$\Gamma^{\mu\nu\sigma}(P_1, P_2) = \langle J^\mu(P_1) J^\nu(P_2) J^\sigma(Q)\rangle$$

Axial anomaly:

$$Q^\sigma\, \Gamma^{\mu\nu\sigma} = -\frac{g^2 N_c}{2}\, \mathrm{tr}(M^3)\quad {}^{\mu\nu\sigma\kappa}\, P_1^\sigma P_2^\kappa$$

In vacuum, need massless mode to satisfy this relation.
Pions do, contribute ~ Q^σ/Q^2.
Otherwise, would have to have massless baryons. *Not* consistent with anomaly

Above shows anomaly does not constrain amplitudes at $T \neq 0$, $\mu \neq 0$.

Especially clear at $\mu \neq 0$: even for massive particles, excitations near Fermi surface have zero energy for ideal system.

Anomalies *will* constrain gaps near Fermi surface: superconductivity...

Banks-Casher and quarkyonic

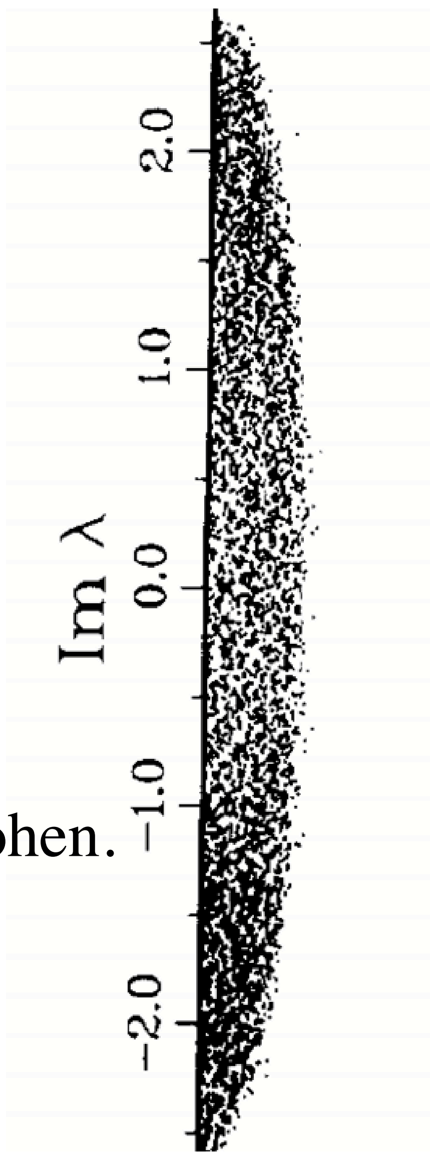
Splittorff & Verbaarschot 0809.5259; Osborn, Splittorff & Verb. 0807.4584...

Use random matrix theory to study the sign problem.
Valid for small box, epsilon regime.
In vacuum, eigenvalues of D purely imaginary:

$$\not{D} = \qquad ; \quad \langle \bar{\ } \ \rangle = \langle \frac{1}{\not{D} + m} \rangle = \int d \quad \frac{(\ )}{\ + m}$$

Banks-Casher: $\langle \bar{\psi} \psi \rangle \sim \rho(0)$

At $\mu \neq 0$, eigenvalues of $D + i\, \mu\, \gamma^0$ complex, spread out into complex plane, as at right.



Epsilon regime shows how to solve “Silver Blaze” problem of Cohen.

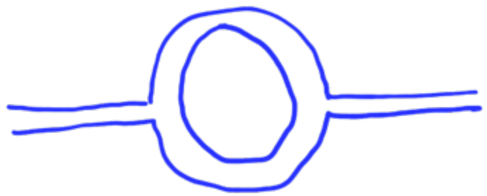
Trivial point: width of distribution determined by μ
Thus for large μ, should obtain chirally symmetric distribution;
i.e., quarkyonic phase

Baryons at Large N_f

Veneziano '78: take *both* N_c and N_f large. Mesons M^{ij} : i,j = 1...N_f .

Thus mesons interact weakly, but there are *many* mesons.

Thus in the hadronic phase, mesons interact *strongly*:



$$\Pi \sim N_f \, g_{3\pi}^2 \sim N_f / N_c$$

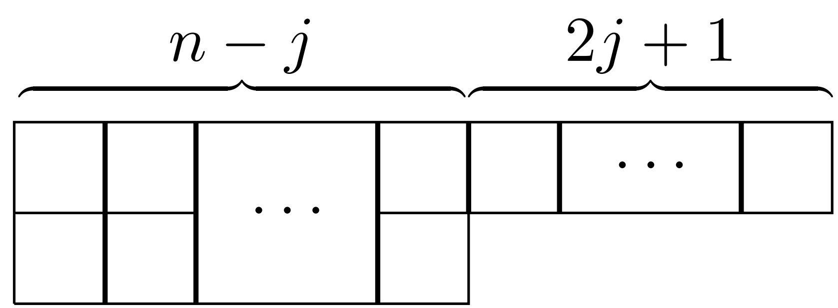
Pressure large in *both* phases:

~ N_f^2 in hadronic phase, ~N_c^2, N_c N_f in "deconfined" phase.

Polyakov loop also nonzero in both phases.

Baryons: lowest state with spin j

has Young tableaux (N_c = 2n + 1) =>



$$d_j = \frac{(2j+2)\,(N_f+n+j)!\,(N_f+n-j-2)!}{(N_f-1)!\,(N_f-2)!\,(n+j+2)!\,(n-j)!}$$

Baryons at Large N_f: order parameters

Y. Hidaka, RDP, & L. McLerran, 0803.0279: Use Sterling's formula,

$$d_j \sim \mathrm{e}^{+N_c\, f(N_f/n)}\ ,\ \ f(x) = (1+x)\log(1+x) - x\log(x)$$

Degeneracy of baryons increases *exponentially*.

Argument is heuristic: baryons are strongly interacting.
Still, difficult to see how interactions can overwhelm exponentially growing spectrum, even for the lowest state.

Use *baryons* as order parameter. At T=0, fluctuations in baryon number,

$\langle B^2 \rangle \neq 0$ when $N_c\, f(N_c/n) = m_B/T$, or

$$T_{qk} = f(N_f/n)\,\frac{m_B}{N_c}$$

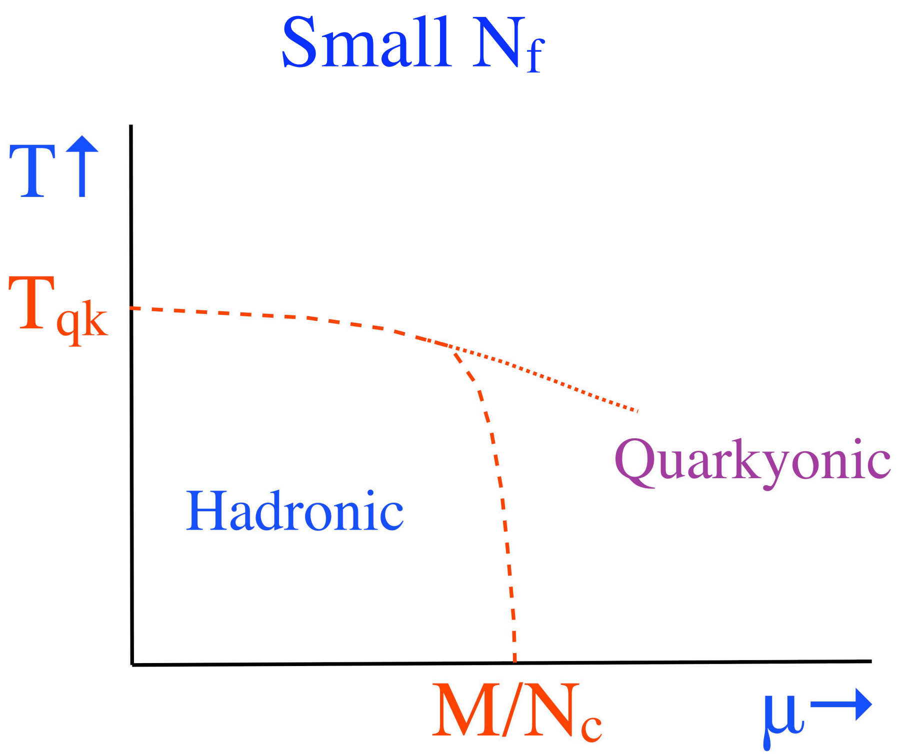
At $\mu \neq 0$, baryon number itself:

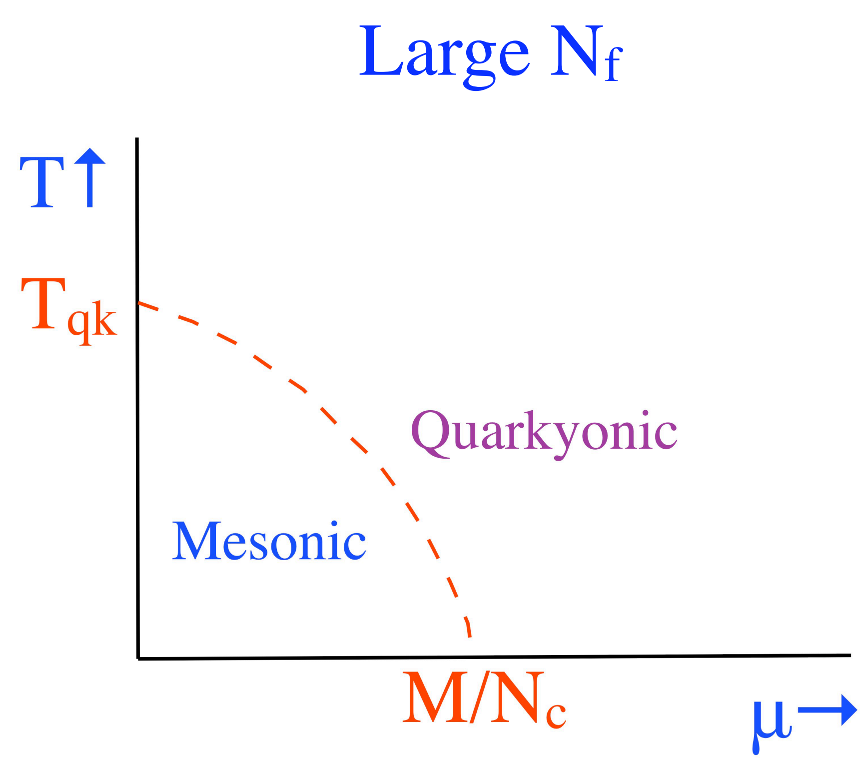
$\langle B \rangle \neq 0$ when $N_c\, f(N_c/n) = (m_B - N_c\, \mu)/T$:

$$T_{qk} = f(N_f/n)\left(\frac{m_B}{N_c} - \mu\right)$$

Possible phase diagrams as N_f increases

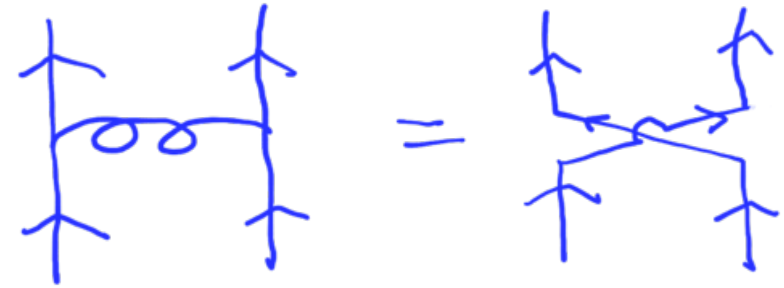
The "rectangle" for small N_f becomes smoothed.
Eventually, maybe the quarkyonic line merges with that for baryon condensation.
All *VERY* qualitative. Clearly many possible phase diagrams!
With SUSY: condensation of Higgs fields as well.





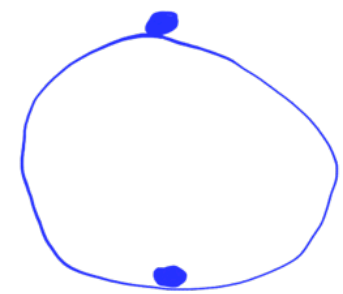
Chiral Density Waves (perturbative)

Excitations near the Fermi surface?



At large N_c, color superconductivity suppressed,
$\sim 1/N_c$: pairing into two-index state:

Also possible to have "chiral density waves", pairing of quark and anti-quark:
Deryagin, Grigoriev, & Rubakov '92. Shuster & Son, hep-ph/9905448.
Rapp, Shuryak, and Zahed, hep-ph/0008207.



Order parameter $\langle\overline{\psi}(-\vec{p}_f)\,\psi(+\vec{p}_f)\rangle$
Sum over color, so *not* suppressed by $1/N_c$.

Pair quark at $+\,p_f$ with anti-quark at $-\,p_f$: for a *fixed* direction.
Breaks chiral symmetry, with state varying $\sim \exp(-\,2\,p_f\,z)$.

Wins over superconductivity in low dimensions. Loses in higher.
Shuster & Son '99: in perturbative regime, CDW only wins for $N_c > 1000\,N_f$

Quarkyonic chiral density waves

Consider meson wave function, with kernel:
Confining potential in 3+1 dimensions like
Coulomb potential in 1+1 dim.s:

$$\int dk_0\, dk_z \int d^2k_\perp \, \frac{1}{(k_0^2 + k_z^2 + k_\perp^2)^2} \sim \int dk_0\, dk_z \, \frac{1}{k_0^2 + k_z^2}$$

In 1+1 dim.'s, behavior of massless quarks near Fermi surface maps ~ $\mu = 0$!

Mesons in vacuum naturally map into CDW mesons.

Witten '84: in 1+1 dim.'s, use non-Abelian bosonization for QCD.

a, b= 1...N_c. i,j = 1... N_f.

$$J_+^{ij} = \overline{\psi}^{a,i} \psi^{a,j} \sim g^{-1} \partial_+ g \; ; \; J_+^{ab} = \overline{\psi}^{a,i} \psi^{b,i} \sim h^{-1} \partial_+ h \; .$$

Steinhardt '80. Affleck '86. Frishman & Sonnenschein, hep-th/920717...
Armoni, Frishman, Sonnenschein & Trittman, hep-th/9805155; AFS, hep-th/0011043..
Bringoltz 0901.4035; Galvez, Hietanan, & Narayanan, 0812.3449.

Bosonized quarkyonic matter

After non-Abelian bosonization, action factorizes into sum of g, in $SU(N_f)$, and h, in $SU(N_c)$. Action for g is

$$8\pi \, S_{WZW} = \int d^2 z \, \mathrm{tr} \, B_i^2 + 2/3 \int d^3 y \, \epsilon^{ijk} \, \mathrm{tr} \, B_i B_j B_k \;, \; B_i = g^{-1} \partial_i g \;.$$

Action for h, is a $SU(N_c)$ gauged WZW model. But: g and h *decouple*!
Spectrum of h complicated, involves massive modes, like usual 't Hooft model.

Spectrum of g is that of usual WZW model, with *massless* modes.

Hence in 1+1 dim.'s, CDW are natural, but with *massless* excitations thereof.

In 3+1 dim.'s: have highly anisotropic state, *somen*-state:
Y. Hidaka, T. Kojo, L. McLerran, & RDP '09...

Chiral condensate ~ Λ_{QCD}^2/μ^2. Length of somen-state large, ~ $\exp(N_c)$.
Quantum fluctuations tend to scramble the somen.

Guess for phase diagram in QCD

Pure guesswork: deconfining & chiral transitions split apart at critical end-point?
Line for deconfining transition first order to the right of the critical end-point?
Critical end-point for deconfinement, or continues down to T=0?