Chiral spirals and their fluctuations

1. Standard phase diagram in $T$ & $\mu$: critical end-point (CEP)

   *Not seen from lattice at small $\mu$*

2. Quarkyonic phase at large $N_c$ (analytic) and $N_c = 2$ (lattice)

3. Chiral Spirals in Quarkyonic matter: sigma models, SU(N) and U(1)

4. Phase diagram: *just a 1st order line,*
   
   *with large fluctuations in the Lifshitz regime*
“Standard” phase diagram for QCD in $T$ & $\mu$: CEP?

Lattice: at quark chemical potential $\mu = 0$, crossover at $T_{ch} \sim 154$ MeV

At $\mu \neq 0$, quarks might change scalar 4-pt coupling $< 0$, so transition 1st order

Must meet at a **Critical End Point (CEP)**, *true* 2nd order phase transition

Asakawa & Yazaki ‘89, Stephanov, Rajagopal & Shuryak ‘98 & ‘99

```
\begin{tikzpicture}
  \draw[->] (0,0) -- (0,6) node[below] {$T$};
  \draw[->] (0,0) -- (6,0) node[right] {$\mu$};
  \draw[red, dashed] (0,6) .. controls (2,2) and (5,2) .. (6,0) node[midway, above] {Quark-Gluon Plasma};
  \draw[red] (0,0) .. controls (2,2) and (5,2) .. (6,0) node[midway, above] {Hadronic};
  \draw[red, dashed] (0,0) .. controls (2,2) and (5,2) .. (6,0) node[midway, above] {crossover};
  \draw[red] (0,0) .. controls (2,2) and (5,2) .. (6,0) node[midway, above] {Critical End Point};
  \draw[red] (0,0) .. controls (2,2) and (5,2) .. (6,0) node[midway, above] {Quark matter};
  \draw[red] (0,0) .. controls (2,2) and (5,2) .. (6,0) node[midway, above] {1st order line};
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Lifshitz phase diagram for QCD

Instead: “Lifshitz regime”: strongly coupled, large fluctuations
Unbroken 1st order line to spatially inhomogeneous phases = “chiral spirals”

Hints in heavy ion data?

Fundamental problem in field theory: analogies to phase diagram for polymers
Could be CEP as well...

[Diagram of the phase diagram with labels for Lifshitz regime, crossover, 1st order line, and chiral spirals.]
Lattice, hot QCD: no CEP at small $\mu$

Lattice: Hot QCD, 1701.04325

Expand about $\mu = 0$, power series in $\mu^{2n}$, $n = 1, 2, 3$.

Estimate radius of convergence. *No* sign of CEP by $\mu_{qk} \sim T$

![Graph showing a plot of $T$ vs $\mu_B$ with a shaded region indicating a disfavored region for the location of a critical point.](image)
So if there is no critical endpoint, what could be going on?
Lattice for $T = 0, \mu \neq 0$, two colors


Heavy pions, $m_\pi \sim 740$ MeV. $\sqrt{\sigma} = 470$ MeV. $32^4$ lattice, $a \sim .04$ fm

Confined until very high $\mu qk \sim 1$ GeV. Bare Polyakov loop:
Lattice for $T = 0$, $\mu \neq 0$, *two* colors

**Lattice:** Bornyakov et al, 1711.01869.

String tension in time: nonzero up to $\mu_{qk} \sim 750$ MeV

![Graph showing the string tension in time as a function of $\mu_{qk}$, with data points indicating a decrease in the ratio $\frac{\sigma_{time}}{\sigma_0}$ for increasing $\mu_{qk}$.](image-url)
Phases for $N_c = 2$, $T \sim 0$, $\mu \neq 0$

Braguta, Ilgenfritz, Kotov, Molochkov, & Nikolaev, 1605.04090 (earlier: Hands, Skellerud + …)

Lattice: $N_c = 2$, $N_f = 2$. $m_\pi \sim 400$ MeV, fixed $T \sim 50$ MeV, vary $\mu_{qk}$.

Hadronic phase: $0 \leq \mu_{qk} < m_\pi / 2 \sim 200$ MeV. Confined, independent of $\mu$

Dilute baryons: $200 < \mu_{qk} < 350$. Bose-Einstein condensate (BEC) of diquarks.

Dense Baryons: $350 < \mu_{qk} < 600$. Pressure not perturbative, BEC

Quarkyonic: $600 < \mu_{qk} < 1100$: pressure $\sim$ perturbative, but excitations confined
(Wilson loop $\sim$ area)

Perturbative: $1100 < \mu_{qk}$, but $\mu a$ too large.
Quarkyonic matter

McLerran & RDP 0706.2191

At large $N_c$, $g^2 N_c \sim 1$, $g^2 N_f \sim 1/N_c$, so need to go to large $\mu \sim N_c^{1/2}$.

$$m_{\text{Debye}}^2 = g^2((N_c + N_f/2)T^2/3 + N_f \mu^2/(2\pi^2))$$

Doubt large $N_c$ applicable at $N_c = 2$.

When does perturbation theory work?

$T = \mu = 0$: scattering processes computable for momentum $p > 1$ GeV

$T \neq 0$: $p > 2\pi T$, lowest Matsubara energy

$\mu \neq 0$, $T = 0$: $\mu$ is like a scattering scale, so perhaps $\mu_{\text{pert}} \sim 1$ GeV.

At least for the pressure. Excitations determined by region near Fermi surface
Possible phases of cold, dense quarks

Confined: $0 \leq \mu_{qk} < \frac{m_{baryon}}{3}$. $\mu$ doesn’t matter

Dilute baryons: $m_{baryon} \frac{3}{\mu_{qk}} < \mu_{qk} < \mu_{dilute}$. Effective models of baryons, pions

Dense baryons: $\mu_{dilute} < \mu_{qk} < \mu_{dense}$. Pion/kaon condensates.

Quarkyonic: $\mu_{dilute} < \mu_{qk} < \mu_{perturbative}$. 1-dim. chiral spirals.

Perturbative: $\mu_{perturbative} < \mu_{qk}$. Color superconductivity

$\mu_{perturbative} \sim 1$ GeV?

Dense baryons and quarkyonic continuously related.

$U(1)$ order parameter in both.
Relevance for neutron stars

Fraga, Kurkela, & Vuorinen 1402.6618.

Maximum $\mu_{qk}$ may reach quarkyonic (for pressure), but true perturbative?

Ghisoiu, Gorda, Kurkela, Romatschke, Säppi, & Vuorinen, 1609.04339: \(\text{pressure}(\mu_{qk}) \sim g^6\).

Will be able to compute $\Lambda_{\text{pert}} = \# \mu_{qk} \# \sim 1$?
Quarkyonic matter: 1-dim. reduction

Kojo, Hidaka, McLerran & RDP 0912.3800: as toy model, assume confining potential

$$\Delta_{00} = \frac{\sigma_0}{(\vec{p}^2)^2}, \quad \Delta_{ij} \sim \frac{1}{p^2}$$

Near the Fermi surface, reduces to effectively 1-dim. problem in patches. For either massless or massive quarks, excitations have zero energy about Fermi surface; just Fermi velocity $v_F < 1$ if $m \neq 0$.

Spin in 4-dim. -> “flavor” in 1-dim., so extended $2N_f$ flavor symmetry,

$$\text{SU}(N_f)_L \times \text{SU}(N_f)_R \rightarrow \text{SU}(2N_f)_L \times \text{SU}(2N_f)_R.$$ Similar to Glozman,1511.05857.

Extended $2N_f$ flavor sym. broken by transverse fluctuations, only approximate.

Number of patches $N_{\text{patch}} \sim \mu/\sigma_0$, so spherical Fermi surface recovered as $\sigma_0 \rightarrow 0$. 
Transitions with # patches

Minimal number of patches = 6.

Probably occurs in dense baryonic phase.

In quarkyonic, presumably weak 1st order transitions as # patches changes.

Like Keplers....
Chiral spirals in 1+1 dimensions

In 1+1 dim., can eliminate $\mu$ by chiral rotation:

$$q' = e^{i\mu z \Gamma_5} q, \quad \bar{q}(\mathcal{D} + i\mu \Gamma_0)q = \bar{q}' \mathcal{D} q', \quad \Gamma_5 \Gamma_z = \Gamma_0$$

Thus a constant chiral condensate automatically becomes a chiral spiral:

$$\bar{q}' q' = \cos(2\mu z) \bar{q} q + i \sin(2\mu z) \bar{q} \gamma_5 q$$

Argument is only suggestive.

N.B.: anomaly ok, gives quark number: $\langle \bar{q} \Gamma_0 q \rangle = \mu/\pi$

Pairing is between quark & quark-hole, both at edge of Fermi sea.
Thus chiral condensate varies in $z$ as $\sim 2 \mu$. 
Bosonization in 1+1 dimensions

Do not need detailed form of chiral spiral to determine excitations.
Use bosonization. For one fermion,

\[ \bar{\psi} \, \partial \psi \leftrightarrow (\partial_i \phi)^2 \]

\( \phi \) corresponds to U(1) of baryon number. In general, non-Abelian bosonization.
For flavor modes,

\[ S_{\text{eff}}^{\text{flavor}} = \int dt \int dz \, \frac{1}{16\pi} \text{tr} (\partial_\mu U^\dagger)(\partial_\mu U) + \ldots \]

where \( U \) is a SU(2 \( N_f \)) matrix.
Do not show Wess-Zumino-Witten terms for level 3 = \# colors.
Also effects of transverse fluctuations, reduce SU(2 \( N_f \)) \rightarrow SU(\( N_f \)); quark mass

Lastly, SU(3) + level 2 \( N_f \) sigma model. Modes are gapped by confinement.
Pion/kaon condensates & U(1) phonon

Overhauser ‘60, Migdal ‘71....Kaplan & Nelson ‘86...

Pion/kaon condensate:

\[ \langle \bar{q}_L q_R \rangle \sim \langle \Phi \rangle \sim \Phi_0 \exp(i(qz + \phi)t_3) \]

Condensate along \( \sigma \) and \( \pi^0 \) => \( t_3 \). Kaon condensate \( \sigma \) and \( K \), etc.

Excitations are the SU(\( N_f \)) Goldstone bosons and a “phonon”, \( \varphi \).

Phases with pion/kaon condensates and quarkyonic Chiral Spirals both spontaneously break U(1), have associated massless field.

Continuously connected: SU(\( N_f \)) of \( \pi/K \) condensate => ~ SU(2 \( N_f \)) of CS’s.

Fluctuations same in both.

Perhaps WZW terms for \( \pi/K \) condensates?
Anisotropic fluctuations in Chiral Spirals

Spontaneous breaking of global symmetry =>
Goldstone Bosons have derivative interactions, $\sim \partial^2$

$\pi/K$ condensates and CS’s break both global and rotational symmetries

Interactions along condensate direction usual quadratic, $\sim \partial_z^2$

Those quadratic in transverse momenta, $\sim \partial_\perp^2$, cancel, leaving quartic, $\sim \partial_\perp^4$.

$$\mathcal{L}_{eff} = f_\pi^2 |(\partial_z - i k_0)U|^2 + \kappa |\partial_\perp^2 U|^2 + \ldots$$

Valid for both the U(1) phonon $\varphi$ and Goldstone bosons $U$

Hidaka, Kamikado, Kanazawa & Noumi 1505.00848;
Lee, Nakano, Tsue, Tatsumi & Friman, 1504.03185; Nitta, Sasaki & Yokokura 1706.02938
No long range order in Chiral Spirals

Consider tadpole diagram with anisotropic propagator

\[ \int d^2 k_\perp dk_z \frac{1}{(k_z - k_0)^2 + (k_\perp)^2} \sim \int d^2 k_\perp \frac{1}{k_\perp^2} \sim \log \Lambda_{\text{IR}} \]

Old story for $\pi/K$ condensates: Kleinert ‘81; Baym, Friman, & Grinstein, ‘82.

Similar to smectic-C liquid crystals: ordering in one direction, liquid in transverse. Hence anisotropic propagator

Nematic phase \hspace{1cm} Smectic phase \hspace{1cm} Cholesteric phase

Increasing opacity
Chiral Spirals in 1+1 dimensions

Overhauser/Migdal’s pion condensate: 

\[(\sigma, \pi^0) = f_{\pi}(\cos(k_0z), \sin(k_0z))\]

Ubiquitous in 1+1 dimensions: Basar, Dunne & Thies, 0903.1868; Dunne & Thies 1309.2443+...

Wealth of exact solutions, phase diagrams at infinite \(N_f\).

Usual Gross-Neveu model:

Phase diagram

\[\langle \bar{q}q \rangle \neq 0\]

Chiral spiral:

\[\langle \bar{q}q \rangle_{CS} \neq 0\]
Chiral Spirals in 3+1 dimensions

In 3+1, common in NJL models: Nickel, 0902.1778 + ....Buballa & Carignano 1406.1367 + ...

In reduction to 1-dim, $\Gamma_{5}^{1\text{-dim}} = \gamma_{0}\gamma_{z}$, so chiral spiral between $\bar{q}q$ & $\bar{q}\gamma_{0}\gamma_{z}\gamma_{5}q$
Both of these phase diagrams are *dramatically* affected by fluctuations:

*no* Lifshitz point in 1+1 or 3+1 dimensions at finite $N$

there *is* a *Lifshitz regime*
Standard phase diagram

\[ \mathcal{L} = (\partial_\mu \phi)^2 + m^2 \phi^2 + \lambda \phi^4 + \kappa \phi^6 \]

Negative quartic coupling, \( \lambda \), turns a 2nd order transition into 1st order. Two phases.

\( \langle \phi \rangle = 0 \)

\( \langle \phi \rangle \neq 0 \)

\( m^2 \rightarrow \)

\( \lambda \uparrow \)

\( \rightarrow 1^{\text{st}} \uparrow \)

\( X = \text{tri-critical point, } m^2 = \lambda = 0 \)
Lifshitz phase diagram (in mean field theory)

\[ \mathcal{L}_{\text{Lifshitz}} = (\partial_0 \phi)^2 + Z (\partial_i \phi)^2 + \frac{1}{M^2} (\partial_i^2 \phi)^2 + m^2 \phi^2 + \lambda \phi^4 \]

Negative kinetic term, \( Z < 0 \), generates spatially inhomogeneous phase, CS. Three phases.

\[ \langle \phi \rangle \neq 0 \quad 1^{\text{st}} \uparrow \quad \langle \phi \rangle = 0 \quad 2^{\text{nd}} \uparrow \quad \langle \phi \rangle_{CS} \neq 0 \]

\( X = \text{Lifshitz point, } m^2 = Z = 0 \)

\( m^2 \rightarrow \)
No massless modes in too few dimensions

No massless modes in $d \leq 2$ dimensions:

$$\int d^2 k \frac{1}{k^2} \sim \log \Lambda_{\text{IR}}$$

Cannot break a continuous symmetry in $d \leq 2$ dimensions: instead of Goldstone bosons, generate a mass *non*-perturbatively.

*Lifshitz point*: $Z = m^2 = 0$, so propagator just $\sim 1/k^4$:

$$\int d^4 k \frac{1}{k^4} \sim \log \Lambda_{\text{IR}}$$

Hence *no* Lifshitz point in $d \leq 4$ (spatial) dimensions.

*Must* generate either a mass $m^2$, *or* term $\sim Z \ p^2 \neq 0$, *non*-perturbatively
Lifshitz regime (shaded):
Z and/or m² are \( \neq 0 \) everywhere
strongly coupled, non-perturbative
Example: inhomogenous polymers

Like mixing oil & water: polymers A & B, with AB diblock copolymer ("co-AB")

Three phases: high temperature, A & B mix, symmetric phase

low temperature, little co-AB: A & B separate, broken phase

c-AB tends to decrease interface tension between A & B phases, can turn it negative. Like Z < 0

Low temperature, high concentration co-AB: "lamellar" phase, stripes of A & B. Like smectic.
Lifshitz point in inhomogenous polymers: mean field

Three phases, symmetric, broken, & spatially inhomogenous

Mean field predicts Lifshitz point at given T & concentration of co-AB
Lifshitz regime in inhomogenous polymers

Instead of Lifshitz point predicted by mean field theory, find
Bicontinuous microemulsion: $Z \neq 0, m^2 = 0$: Lifshitz regime
Bicontinuous microemulsion: $Z \approx 0$

Experiment
Jones & Lodge,
Polymer Jour. 131 (44) 2012

Self-consistent field theory
Fredrickson, “The equilibrium theory of inhomogenous polymers”
Phase diagram for QCD in $T$ & $\mu$: usual picture

Two phases, one Critical End Point (CEP) between crossover and line of 1$^{\text{st}}$ order transitions

Ising fixed point, dominated by *massless* fluctuations at CEP
Lifshitz phase diagram for QCD

Lifshitz regime: strongly coupled, large fluctuations
Unbroken 1st order line to spatially inhomogeneous phases = “chiral spirals”

Heavy ions: could go through two 1st order transitions
$T_0$: maximum $T$, point of equal concentrations (unequal entropy)
Fluctuations at 7 GeV

Beam Energy Scan, down to 7 GeV.
Fluctuations $MUCH$ larger when up to 2 GeV than to 0.8 GeV
Trivial multiplicity scaling? ... or Chiral Spiral?
But fluctuations in nucleons, not pions.
X. Luo & N. Xu, 1701.02105, fig. 37; Jowazee, 1708.03364

\[ c_n = \frac{\partial n}{\partial \mu n} p(T, \mu) \]

\[ \frac{c_4}{c_2} \uparrow \]

\[ \uparrow \text{to } 0.8 \text{ GeV} \]
Experimentally

For *any* sort of periodic structure (1D, 2D, 3D...),

Fluctuations concentrated about some characteristic momentum $k_0$

So “slice and dice”: bin in intervals, 0 to .5 GeV, .5 to 1., etc.

*If* peak in fluctuations in a bin not including zero,
*may* be evidence for $k_0 \neq 0$.

*Signals for Lifshitz regime?*

*Must measure fluctuations in pions, kaons...*
NJL models and Lifshitz points

Consider Nambu-Jona-Lasinio models.
Nickel, 0902.1778 & 0906.5295 + .... + Buballa & Carignano 1406.1367

\[ \mathcal{L}_{\text{NJL}} = \overline{\psi}(\partial + g\sigma)\psi + \sigma^2 \]

Integrating over \( \psi \),

\[ \text{tr} \log(\partial + g\sigma) \sim \ldots + \kappa_1((\partial\sigma)^2 + \sigma^4) + \ldots \]

Due to scaling, \( \partial \rightarrow \lambda \partial, \sigma \rightarrow \lambda \sigma \).
Consequently, in NJL @ 1-loop, \textit{tricritical} = \textit{Lifshitz point}.

Special to including only \( \sigma \) at one loop.
Not generic: violated by the inclusion of more fields, to two loop order, etc.

Improved gradient expansion near critical point:
Carignano, Anzuni, Benhar, & Mannarelli, 1711.08607.
Symmetric to CS: 1D (Brazovski) fluctuations

Consider $m^2 > 0, Z < 0$: minimum in propagator at nonzero momentum

Brazovski ‘75; Hohenberg & Swift ‘95 + ...;
Lee, Nakano, Tsue, Tatsumi & Friman, 1504.03185; Yoshiike, Lee & Tatsumi 1702.01511

$$\Delta^{-1} = m^2 + Z k^2 + k^4 / M^2$$
$$= m_{\text{eff}}^2 - 2 Z k_z^2 + O(k_z^3, k_z k_{\perp}^2)$$

$k=(k_{\perp}, k_z - k_0)$: no terms in $k_{\perp}^2$, only $(k_{\perp}^2)^2$.

Due to spon. breaking of rotational sym.

1-loop tadpole diagram:

$$\int d^3k \frac{1}{k_z^2 + m_{\text{eff}}^2 + \ldots} \sim M^2 \int \frac{dk_z}{k_z^2 + m_{\text{eff}}^2} \sim \frac{M^2}{m_{\text{eff}}}$$

Effective reduction to 1-dim for any spatial dimension d, any global symmetry
1\textsuperscript{st} order transition in 1-dim.

*Strong* infrared fluctuations in 1-dim., both in the mass:

\[ \Delta m^2 \sim \lambda \int d^3k \frac{1}{k_z^2 + m_{\text{eff}}^2 + \ldots} \sim \lambda \frac{M}{m_{\text{eff}}} \]

and for the coupling constant:

\[ \Delta \lambda \sim -\lambda^2 \int \frac{d^3k}{(k_z^2 + m_{\text{eff}}^2 + \ldots)^2} \sim -\lambda^2 M^3 \int m_{\text{eff}} \frac{dk_z}{k_z^4} \sim -\lambda \frac{M^3}{m_{\text{eff}}^3} \]

*Cannot* tune $m_{\text{eff}}^2$ to 0: $\lambda_{\text{eff}}$ goes negative, 1\textsuperscript{st} order trans. induced by fluctuations

*Not* like other 1st order fluc-ind’d trans’s: just that in 1-d, $m_{\text{eff}}^2 \neq 0$ always