Structure of the Configuration space

We want to analyze the structure of the configuration space for gauge theories for which all particles (states) are in $\mathbb{Z}_N$- invariant representations of the gauge group $SU(N)$. Very often, the statement that gauge group is $SU(N)/\mathbb{Z}_N$ is used to define (characterize) this situation. But for our analysis it is better to proceed in a slightly different way.

First of all, we take $A_0=0$ and analyze the configuration space relevant for a Hamiltonian analysis. The relevant spaces are then $\mathcal{A}$, the space of all 3-dim gauge potentials $A_i(\vec{x})$, and $\mathcal{G}_*$, the space of all gauge transformations. For the latter space, consider

$$\mathcal{G} = \{ g(\vec{x}) : \mathbb{R}^3 \to SU(N) \} \tag{1}$$

$g(\vec{x}) \to g_\infty \in SU(N)$ at spatial infinity. We assume g_∞ is a constant, not necessarily $\mathbb{1}$, which is independent of angular ~~directions~~ variables as $|\vec{x}| \to \infty$. The possible values of g_∞ form a copy of $SU(N)$. Now define

$$\tilde{\mathcal{G}} = \{ g(\vec{x}) : \mathbb{R}^3 \to SU(N) \text{ such that } g \to 1 \text{ as } |\vec{x}| \to \infty \} \tag{2}$$

Evidently $\mathcal{G}/\tilde{\mathcal{G}} = SU(N)$, the set of (rigid) global transformations.

From the closure condition for gauge transformations and the

Gauss law, we know that the wave functions ψ are invariant under $\tilde{\mathcal{G}}$. The global transformations $\mathcal{G}/\tilde{\mathcal{G}}$ act as a Noether symmetry generating states which are multiplets of $SU(N)$. These are the charged states of the theory. (If we have confinement, such states do not exist and the full $\mathcal{G}$ acts trivially on physical states. But that is to dynamically determined. We cannot impose $\mathcal{G}$-invariance at this stage.)

The $\mathbb{Z}_N$-transformations are necessarily global since there is no small deformation of an element of $\mathbb{Z}_N$. ~~They belong to~~ The $\mathbb{Z}_N$-elements act trivially on the physical states of the theory since we have only rep.'s of $SU(N)/\mathbb{Z}_N$ for the charged states. Therefore, in addition to $\tilde{\mathcal{G}}$, we can impose $\mathbb{Z}_N$-invariance on ψ's. Hence we define

$$\mathcal{G}_* = \left\{ g(\vec{x}) : \mathbb{R}^3 \to SU(N) \text{ such that } g \to \omega \in \mathbb{Z}_N \text{ as } |\vec{x}| \to \infty \right\}. \tag{3}$$

This is the essential point: In a theory with $\mathbb{Z}_N$-invariant states, the group of gauge transformations is $\mathcal{G}_*$, not $\tilde{\mathcal{G}}$.

We now turn to $\mathcal{A}$. This is well-known to be an affine space with any two points connected by a straight line. (If $A_i^{(1)}(x)$, $A_i^{(2)}(x)$ are gauge potentials, $A_i^{(1)}(x)\,\tau + (1-\tau)A_i^{(2)}(x)$ is also a gauge potential. ~~with end-points~~) Nontrivial topology arises upon quotienting by $\mathcal{G}_*$. Consider a sequence of potentials given by

$$A_i(\vec{x},\tau) = A_i^g(\vec{x})\,\tau + (1-\tau)\,A_i(\vec{x}) \tag{4}$$

$$A_i^g = g^{-1} A_i\, g + g^{-1}\partial_i g \tag{5}$$

The end-points of this line in configurations space are gauge transforms of each other if $g(\vec{x})$ is an element of $\mathcal{G}_*$. Thus it is a closed loop. If such a loop is contractible, we can write

$$A_i(\vec{x},\tau) = g^{-1}(\vec{x},\tau)\, A_i(\vec{x})\, g(\vec{x},\tau) + g^{-1}(x,\tau)\,\partial_i g(x,\tau)$$

with $g(\vec{x},0) = 1$, $g(\vec{x},1) = g(\vec{x})$. Hence $g(\vec{x})$ is a smooth deformation of homotopically equivalent to 1. If we choose $g(\vec{x})$ to be homotopically nontrivial, the loop is not contractible in $\mathcal{A}/\mathcal{G}_*$. $A_i(x,\tau)$ has nonzero instanton number. (One can make better ~~choices~~ interpolations which minimize the action giving the selfdual instantons.)

Normally the above analysis leads to the usual instantons. In our case, $\mathcal{G}_*$ is larger since the asymptotic behaviour is more general. Since the behaviour at infinity is modified, with transformations approaching $w \in \mathbb{Z}_N$ allowed, the simplest way to construct such "instantons" is to use the set of configurations $U^{-1}A_iU + U^{-1}\partial_i U$, where $U = U(\vec{x},\tau)$. These are not gauge transforms of A_i since we do not require that $U \to 1$ (or w) at ∞ for all τ. The instanton number is easily calculated.

$$A^u = U^{-1} A U + U^{-1} dU \tag{6}$$

$$dA^u + A^{u2} = U^{-1}(dA + A^2) U + dt \frac{\partial}{\partial t}(U^{-1} A U + U^{-1} dU)$$

$$= U^{-1} F U + dt\left[-U^{-1}\dot{U}\, U^{-1} A U + U^{-1} A \dot{U} - U^{-1}\dot{U}\, U^{-1} dU + U^{-1} d\dot{U}\right]$$

$$= U^{-1}\left[F + d(\dot{U}U^{-1}) + A\dot{U}U^{-1} - \dot{U}U^{-1}AU\right]U$$

$$= U^{-1}(F + dt\, Da)\, U$$

$$= U^{-1}(F - Da)\, U \tag{7}$$

$$\mathcal{V} = -\frac{1}{8\pi^2}\int \mathrm{Tr}\,(F - Da)(F - Da)$$

$$= +\frac{1}{4\pi^2}\int \mathrm{Tr}(Da\, F) = \oint \frac{\mathrm{Tr}(a F)}{4\pi^2} \tag{8}$$

We need nonzero flux for F at spatial ∞ for this to contribute $a = \dot{U}U^{-1}$. The time-integral should give the change in U ~~as~~ from $U = 1$ at $\tau = 0$ to $U = \omega$ at $\tau = 1$. In the example we considered $F \longrightarrow i\frac{\sigma\cdot\hat{x}}{4} d\Omega$ as $|\vec{x}| \to \infty$ and $U = \exp\left[i\int_0^\tau \sigma\cdot\hat{x}\,\frac{\pi f}{\beta}\, d\tau\right]$, $f \to 1$ as $|\vec{x}| \to \infty$.

$$\mathcal{V} = \frac{1}{4\pi^2}\int \mathrm{Tr}\left(i\,\sigma\cdot\hat{x}\,\frac{\pi}{\beta}\right) i\frac{\sigma\cdot\hat{x}}{4}\, d\Omega\, d\tau$$

$$\nu = -\frac{1}{16\pi^2}\, 2 \times 4\pi \int_0^\beta \frac{d\tau\, \pi}{\beta}$$
$$= -\tfrac{1}{2}. \tag{9}$$

The asymptotic behaviour of U is $U \to -1$ as $|\vec{x}| \to \infty$. So this describes a closed loop for the $SU(2)/\mathbb{Z}_2 = SO(3)$ theory. For $SU(N)/\mathbb{Z}_N$, we expect $\nu = 1/N$ + integer. The basic argument for this is as follows. The elementary loop in $SU(N)$-theory has $\nu = 1$ and in this case $U \to 1$ as $|\vec{x}| \to \infty$, $\tau = \beta$. Such a loop can be made by N traversals of the ~~loop~~ ~~with~~ ~~a~~ loop in $SU(N)/\mathbb{Z}_N$, since $U \to \exp\left(\frac{2\pi i}{N}\right)$ for the elemental loop in $SU(N)/\mathbb{Z}_N$. Thus $\nu_{SU(N)/\mathbb{Z}_N} \cdot N = \nu_{SU(N)}$.

There is another issue we need to address in connection with this fractional value of ν. The instanton number occurs in the index theorem. How can we have fractional number of zero modes? Since all fields are in $\mathbb{Z}_N$-invariant representations, the relevant index is proportional to $Tr(F^2)_R$. The trace is evaluated in a rep. R which is $\mathbb{Z}_N$-invariant. Since $Tr(t^a t^b)_R = A_R\, Tr(t^a t^b)_F = \frac{1}{2} A_R \delta^{ab}$, we find that

$$\text{index } \not{D} = A_R\, \nu \tag{10}$$

$$A_R = \frac{2\, C_R \dim R}{\dim G}, \quad C_R = \text{quadratic Casimir for } R.$$

A_R is an integral multiple of N for all Z_N-invariant rep.'s of $SU(N)$. Therefore there will be no contradictions with the index theorem.

We now consider closed noncontractible loops starting from the vacuum. Take $A=0$ as a representative potential for the vacuum. (classical vacuum) The final point in $\mathcal{A}$ is the pure gauge configuration $g^{-1}dg$. The intermediate configurations (or the path) may be taken as, (eqn.(4) with $A_i=0$),

$$A = \tau\, g^{-1}dg \tag{11}$$

$$dA + A^2 = d\tau\, g^{-1}dg - \tau (g^{-1}dg)^2 + \tau^2 (g^{-1}dg)^2.$$

$$\begin{aligned} \nu &= -\frac{1}{8\pi^2}\int (d\tau\, g^{-1}dg)(\tau^2-\tau)(g^{-1}dg)^2 \cdot 2 \\ &= -\frac{1}{4\pi^2}\cdot \int \mathrm{Tr}(g^{-1}dg)^3 \cdot \left(\tfrac{1}{3}-\tfrac{1}{2}\right) \\ &= \frac{1}{24\pi^2}\int \mathrm{Tr}(g^{-1}dg)^3 = Q \end{aligned} \tag{12}$$

Q = winding number for $S^3 \to G$ maps. Since Q is an integer, ν has to be an integer. Thus vacuum $\to$ vacuum transitions will always see an integer value for ν. This implies that lattice simulations with periodic boundary conditions will always see N copies of the fractional instantons.

Energy considerations

Write $dr = \frac{1}{2r}(2x_i dx_i) = \frac{x_i}{r} dx_i$

$\Rightarrow$

$$F = -\frac{i}{2r} K' \epsilon_{abc} \sigma_a \hat{x}_c \hat{x}_k \, dx^k \wedge dx^b$$

$$+ i\left(\frac{1-K^2}{4}\right) \sigma_a \hat{x}_a \, \frac{\epsilon_{ijk} \hat{x}_k}{r^2} dx^i dx^j .$$

$$F^a_{mn} = \frac{i}{r} K' \left[\epsilon_{anc} \hat{x}_c \hat{x}_m - \epsilon_{amc} \hat{x}_c \hat{x}_n\right]$$

$$- i \hat{x}^a \left(\frac{1-K^2}{2}\right) \epsilon_{mnk} \frac{\hat{x}_k}{r^2}$$

$$= \frac{K'}{r}\left(\epsilon_{anc} \hat{x}_c \hat{x}_m - \epsilon_{amc} \hat{x}_c \hat{x}_n\right)$$

$$- \left(\frac{1-K^2}{2}\right) \frac{\hat{x}^a \hat{x}^k}{r^2} \epsilon_{mnk} .$$

$$Da = -i \frac{\sigma_a}{2} \hat{x}_a \left(\frac{2\pi}{\beta} f'\right) \hat{x}_m dx^m d\tau$$

$$- \frac{iK}{2r} \sigma_a \, d\hat{x}^a \wedge \frac{2\pi f}{\beta} d\tau$$

$$+ i \frac{K}{2r}\left(\frac{2\pi f}{\beta}\right) \sigma_a \hat{x}_a \hat{x}_b \, dx^b \wedge d\tau$$

$$E^a_m = \hat{x}_a \left(\frac{2\pi f'}{\beta}\right) \hat{x}_m + \frac{K}{r} \delta^a_m \left(\frac{2\pi f}{\beta}\right) - \frac{K}{r}\left(\frac{2\pi f}{\beta}\right) \hat{x}^a \hat{x}_m$$

$$= \frac{Kf\,2\pi}{\beta r} \delta^a_m + \hat{x}^a \hat{x}_m \left(\frac{2\pi f'}{\beta} - \frac{K}{r}\frac{2\pi f}{\beta}\right)$$

<u>Calculate E^2:</u>

$$E^a_m E^a_m = \left(\frac{2\pi}{\beta}\right)^2 \left(\frac{Kf}{r}\right)^2 \times 3 + 2\left(\frac{2\pi}{\beta}\right)\left(\frac{Kf}{r}\right)\left(\frac{2\pi f'}{\beta} - \frac{Kf}{r}\frac{2\pi}{\beta}\right) + \left(\frac{2\pi}{\beta}\right)^2 \left(f' - \frac{Kf}{r}\right)^2$$

$$= \left(\frac{2\pi}{\beta}\right)^2 \left[3\left(\frac{Kf}{r}\right)^2 + 2\frac{Kf}{r}\left(f' - \frac{Kf}{r}\right) + f'^2 - 2f'\frac{Kf}{r} + \left(\frac{Kf}{r}\right)^2 \right]$$

$$= \left(\frac{2\pi}{\beta}\right)^2 \left[f'^2 + 2\left(\frac{Kf}{r}\right)^2 \right]$$

$$F^a_{mn} F^a_{mn} = \left[\frac{K'}{r} \epsilon_{a\underset{\cdot}{m}c} \hat{x}_c \hat{x}_{\underset{\cdot}{n}} + \left(\frac{1-K^2}{2}\right) \frac{\epsilon_{mnk}}{r^2} \hat{x}_c \hat{x}^a\right]^2$$

$$= \left(\frac{K'}{r}\right)^2 \epsilon_{a\underset{\cdot}{m}c} \hat{x}_c \hat{x}_{\underset{\cdot}{n}} \epsilon_{a\underset{\cdot}{m}k} \hat{x}_k \hat{x}_{\underset{\cdot}{n}}$$

$$+ \cancel{\frac{2K'}{r}\left(\frac{1-K^2}{2}\right)\frac{1}{r^2}\hat{x}_k \epsilon_{mnk}}$$

$$+ \left(\frac{1-K^2}{2}\right)^2 \cancel{\epsilon_m} \frac{2}{r^4}$$

$$= \left(\frac{1-K^2}{2}\right)^2 \cdot \frac{2}{r^4} + \left(\frac{K'}{r}\right)^2 2\, \epsilon_{amc} \hat{x}_c \hat{x}_n \left(\epsilon_{amk} \hat{x}_k \hat{x}_n - \epsilon_{ank} \hat{x}_k \hat{x}_m\right)$$

$$= \frac{(1-K^2)^2}{2} \frac{1}{r^4} + 2\left(\frac{K'}{r}\right)^2 \Big[2 - \underbrace{\delta^{mc}_{nk} \hat{x}_c \hat{x}_k \hat{x}_m \hat{x}_n}_{0}\Big]$$

$$= 4\left(\frac{K'}{r}\right)^2 + \frac{(1-K^2)^2}{2} \frac{1}{r^4}$$

The Hamiltonian is given by

$$\mathcal{H} = \int d^3x \, \frac{1}{2}\left[E^a_m E^a_m + \frac{1}{2} F^a_{mn} F^a_{mn}\right]$$

$$= \frac{1}{2}\int d^3x \left[\left(\frac{2\pi}{\beta}\right)^2 \left\{f'^2 + 2\left(\frac{Kf}{r}\right)^2\right\} + 2\left(\frac{K'}{r}\right)^2 + (1-K^2)^2 \frac{1}{4r^4}\right]$$

Variational Ansatz

Behaviour ~~of f&K~~: $f(r=\infty) = 1$

$$Da = -\frac{i\sigma^a}{2}\,\hat{x}^a\left(\frac{2\pi}{\beta}\right)\frac{\partial f}{\partial r}\,\hat{x}^b\,dx^b\wedge d\tau - \frac{i}{2}\sigma^a\,\frac{K}{r}\left(\delta^{ab}-\hat{x}^a\hat{x}^b\right)dx^b\wedge\alpha$$

$$E^a_m = \frac{2\pi}{\beta}\,\frac{Kf}{r}\,\delta^a_m + \frac{2\pi}{\beta}\,\hat{x}^a\hat{x}_m\left(f' - \frac{Kf}{r}\right) \qquad (1)$$

Near $r=0$, $K\to 1$. $\frac{f}{r}$ and f' should vanish to avoid singularity due to $\hat{x}^a\hat{x}^m$. Thus $f(r)\sim r^{\epsilon+1}$ near $r=0$

$$F^a_{mn} = -\frac{K'}{r}\left(\epsilon_{amc}\,\hat{x}_m\hat{x}_n - m\leftrightarrow n\right) - \left(\frac{1-K^2}{2}\right)\frac{\epsilon_{mnc}\,\hat{x}^a\hat{x}^c}{r^2}$$

We need $K(\infty)=0$, $K(0)=1$. To avoid singularity due to $\hat{x}^a\hat{x}^c$, $1-K^2 \simeq 2(1-K)\sim r^{2+\epsilon}$.
$K \approx 1 + r^{2+\epsilon}$, In this case $\frac{K'}{r}\sim r^{\epsilon}$ also, so this is okay for both terms.

$$f = \begin{cases} 1 & \text{as } r\to\infty \\ r^{1+\epsilon} & \text{as } r\to 0 \end{cases}$$

$$K = \begin{cases} 0 & \text{as } r\to\infty \\ 1+r^{2+\epsilon} & \text{as } r\to 0 \end{cases}$$

Actually we can do a little better.

$$F^a_{mn} = -\epsilon_{mna}\frac{K'}{r} + \left[\frac{K'}{r} - \frac{1-K^2}{2}\right]\epsilon_{mnk}\hat{x}^k\hat{x}^a$$

So, near $r=0$,

$$\frac{K'}{r} - \frac{1-K^2}{2} \approx \left[\frac{K'}{r} - (1-K)\right] \to 0 \quad \text{if } K = 1+\lambda r^2$$

This is the case of the monopole.

Similar reasoning applies to f. $f' - \frac{f}{r}$ vanishes for $f \approx r$. So the requirements are

$$f = \begin{cases} 1 & \text{as } r \to \infty \\ r & \text{as } r \to 0 \end{cases}$$

$$K = \begin{cases} 0 & \text{as } r \to \infty \\ 1+r^2 & \text{as } r \to 0 \end{cases} \qquad (2)$$

We can choose a BPS ansatz with the scale parameter as the variational parameter.

$$K = \frac{mr}{\sinh mr} \qquad f = \tanh mr \qquad (3)$$

More generally, take

$$K = \frac{\lambda r}{\sinh \lambda r} \qquad f = \tanh mr \qquad (4)$$

with two parameters λ, m.

Estimate of energy

The electric field E we used is just $\dot{A}$. Thus, including the coupling constant e, the energy is given by

$$\mathcal{E} = \frac{1}{2e^2}\int d^3x \left[\frac{4\pi^2}{\beta^2} f'^2 + \frac{8\pi^2}{\beta^2}\left(\frac{Kf}{r}\right)^2 + 2\left(\frac{K'}{r}\right)^2 + \frac{(1-K^2)^2}{4r^2}\right] \quad (5)$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}$$

$f = \tanh mr \qquad f' = m\,\mathrm{sech}^2 mr.$

$$\mathrm{I} = \frac{2\pi^2}{e^2\beta^2}\cdot m^2 \int dr\; r^2 (\mathrm{sech}^2 mr)\; 4\pi$$

$$= \frac{8\pi^3}{e^2\beta^2}\,\frac{1}{m}\int_0^\infty du\; u^2\,\mathrm{sech}^2 u$$

$$= \frac{8\pi^3}{e^2\beta^2}\,\frac{1}{m}\; J_1 \quad (6)$$

$$J_1 = \int_0^\infty du\; u^2\,\mathrm{sech}^2 u \simeq 0.215 \qquad \text{(numerically)}$$

$$\mathrm{II} = \frac{4\pi^2}{e^2\beta^2}\cdot 4\pi \int_0^\infty dr\; \frac{r^2\;\lambda^2}{\sinh^2\lambda r}\cdot \tanh^2 mr$$

$$= \frac{16\pi^3}{e^2\beta^2}\,\frac{1}{\lambda}\int_0^\infty du\; u^2\,\frac{\tanh^2 au}{\sinh^2 u}$$

$$\mathrm{II} = \frac{16\pi^3}{e^2\beta^2}\,\frac{1}{\lambda}\quad J_2(a) \tag{7}$$

$$J_2(a) = \int_0^\infty du\; u^2\,\frac{\tanh^2 au}{\sinh^2 u} \tag{8}$$

$a = m/\lambda$.

$$\mathrm{III} = \frac{1}{e^2}\int d^3x\,\left(\frac{K'}{r}\right)^2$$

$$= \frac{1}{e^2}\,4\pi\int dr\; r^2\,\frac{1}{r^2}\cdot\left(\frac{\lambda}{\sinh\lambda r}\right)^2\left[1 - \lambda r\coth\lambda r\right]^2$$

$$= \frac{4\pi\lambda}{e^2}\int_0^\infty du\;\frac{(1 - u\coth u)^2}{\sinh^2 u}$$

$$= \frac{4\pi\lambda}{e^2}\;J_3 \tag{9}$$

$$J_3 = \int_0^\infty du\;\frac{(1 - u\coth u)^2}{\sinh^2 u} \simeq 0.215 \tag{10}$$

$$\mathrm{IV} = \frac{1}{8e^2}\;4\pi\int_0^\infty dr\; r^2\left[1 - \frac{(\lambda r)^2}{\sinh^2\lambda r}\right]^2\frac{1}{r^4}$$

$$= \frac{\pi\lambda}{2e^2}\int_0^\infty du\;\frac{1}{u^2}\left(1 - \frac{u^2}{\sinh^2 u}\right)^2$$