A new approach to inclusive decay spectra

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Plan of the talk

• Introduction: inclusive $B$ decay spectra — the challenge

• Sudakov resummation with NNLL accuracy

• Divergence of perturbation theory

• The quark distribution in the meson — cancellation of renormalon ambiguities

• Dressed Gluon Exponentiation: renormalon resummation in the Sudakov exponent

• computed $\bar{B} \longrightarrow X_s \gamma$ spectrum — comparison to data
Inclusive B–decay Spectra and Infrared Renormalons

References

• Taming the $\bar{B} \rightarrow X_s \gamma$ spectrum by Dressed Gluon Exponentiation, J.R. Andersen, E. Gardi, accepted for publication in JHEP [hep-ph/0502159].

• On the quark distribution in an on-shell heavy quark and its all-order relations with the perturbative fragmentation function, E. Gardi, JHEP 0502, 053 (2005) [hep-ph/0501257].

Motivation: flavor physics

Identifying deviations from SM by

- (over)constraining the unitarity triangle
- branching ratios of rare decays

Decays are difficult to compute since quarks are confined into mesons

- both exclusive decays (specific final states) and inclusive ones (e.g. $\bar{B} \rightarrow X_c l \bar{\nu}$, $\bar{B} \rightarrow X_u l \bar{\nu}$, $\bar{B} \rightarrow X_s \gamma$) are useful.

But inclusive decays are insensitive to the details of the final and initial states. In particular, the absolute inclusive rate is computable in perturbation theory.
Hierarchy of scales and the Operator Product Expansion

separation between short– and long–distance physics

First step: integrate out fields with virtuality $\mathcal{O}(M_W)$

$$H = \sqrt{2} G_F V_{tb} V_{ts}^* A \left( \frac{m_t^2}{m_W^2} \right) \frac{e m}{16 \pi^2} \bar{s} L \sigma^{\mu \nu} b_R F_{\mu \nu}$$

To leading order in $1/m_W$ and in $\alpha_s$ (only $O_7$):

Second step: integrate out fields with virtuality $\mathcal{O}(m_b)$

$m_b \gg \Lambda \Rightarrow$ the $b$ quark is close to its mass shell

Heavy–Quark Effective Theory

Expansion in $\Lambda/m_b$ is useful for the calculation of total rates but insufficient for spectra.
Inclusive B–decay Spectra

radiative decay: \( \bar{B} \rightarrow X_s \gamma \)

semi-leptonic decay: \( \bar{B} \rightarrow X_u l \bar{\nu}_l \)

The distribution peaks close to the endpoint \( (E_\gamma \rightarrow M_B/2; \text{ small } M_X) \)

Example: extracting \( |V_{ub}| \) from the semi-leptonic decay

Precise measurements are restricted to the small \( M_X \) region (charm background)

Determination of \( |V_{ub}| \) relies on calculation of the spectrum.
Kinematics in $B \to X_s \gamma$

Most of the B meson momentum is carried by the b quark

$$x \equiv \frac{2E_\gamma}{m_b}, \quad \left. \frac{1}{\Gamma_{\text{tot}}} \frac{d\Gamma}{dx} \right|_{\text{LO}} = \delta(1-x)$$

Beyond LO: the peak is smeared.

Perturbative endpoint: $x = 1$

The quark distribution in the B meson $f(z; \mu)$

In the $B$ rest frame all components of $k$ are $\mathcal{O}(\Lambda) \ll m_b$

so for sufficiently inclusive measurement

the quark distribution effect is power suppressed

But near the endpoint the effect of $f(z; \mu)$ is $\mathcal{O}(1)$

$$m_X^2 = (k + p - q)^2 \simeq (p - q)^2 + 2k \cdot (p - q) = \mathcal{O}(\Lambda M)$$

Neubert; Bigi, Shifman, Uraltsev & Vainshtein (93)
Large–$x$ factorization in inclusive $B$ decays

The spectrum can be computed in PT: infrared– and collinear–safe

Dominated by Sudakov logs, $\ln(1-x)$

**scales:**
- **Hard:** $m$
- **Jet:** $m^2_x = (P_b - q)^2 \simeq m^2(1-x) \implies m^2/N$
- **Soft:** $m(1-x) \implies m/N$

**Spectral moments:**

$$\Gamma_{N}^{\text{PT}} \equiv \int_{0}^{1} dx x^{N-1} \frac{1}{\Gamma_{\text{tot}}^{\text{PT}}} d\Gamma_{\text{PT}}^{\text{tot}}$$

$$= H(m) J(m^2/N; \mu) S_{\text{PT}}(m/N; \mu) + O(1/N)$$

$$\equiv H(m) \text{Sud}(N, m) + O(1/N)$$
Coefficients in the Sudakov exponent

\[ \text{Sud}(N, m) = \exp \left\{ - \sum_{n=1}^{\infty} \sum_{k=1}^{n+1} C_{n,k} \ln^k N \left( \frac{\alpha_s^\text{MS}(m^2)}{\pi} \right)^n \right\} \]

The coefficients \( C_{n,k} \) are known exactly to NNLL accuracy. For \( N_f = 4 \) they are:

\[
\begin{array}{cccccccc}
 n & -1.564 & 0.667 & 0 & 0 & 0 & 0 & 0 \\
 k & 3.837 & -0.078 & 1.389 & 0 & 0 & 0 & 0 \\
 & ? & 20.579 & 6.339 & 3.376 & 0 & 0 & 0 \\
 & ? & ? & 116.464 & 33.024 & 9.042 & 0 & 0 \\
\end{array}
\]

• At a given order in \( \alpha_s \) the coefficients of subleading logs (lower \( k \)) get large...

• Is the fixed-logarithmic-accuracy approximation at LL / NLL / NNLL good?
Conventional Sudakov resummation with NNLL accuracy

\[
\text{Sud}(N, m) = \exp \left\{ \sum_{n=0}^{\infty} g_n(\lambda) \left( \frac{\alpha_s^\text{MS}(m^2)}{\pi} \right)^{n-1} \right\}; \quad \lambda \equiv \frac{\alpha_s^\text{MS}(m^2)}{\pi} \beta_0 \ln N
\]

\[
g_0(\lambda) = \frac{C_F}{\beta_0^2} \left[ (1 - \lambda) \ln (1 - \lambda) - \frac{1}{2} (1 - 2\lambda) \ln (1 - 2\lambda) \right]
\]

**Corresponding spectra**
Coefficients in the Sudakov exponent in the large–$\beta_0$ limit

$$\text{Sud}(N, m) = \exp\left\{-\sum_{n=1}^{\infty} \sum_{k=1}^{n+1} C_{n,k} \ln^k N \left(\frac{\alpha_s^{\overline{\text{MS}}}(m^2)}{\pi}\right)^n\right\}$$

The part in $C_{n,k}$ that is proportional to $(\beta_0)^{n-1}$ is known to all orders:

\[ \begin{array}{cccccccc}
    n & -1.56 & 0.67 & 0 & 0 & 0 & 0 & 0 \\
    1.24 & 0.90 & 1.39 & 0 & 0 & 0 & 0 & 0 \\
    61.17 & 28.32 & 8.28 & 3.38 & 0 & 0 & 0 & 0 \\
    1096.06 & 515.20 & 166.25 & 34.89 & 9.04 & 0 & 0 & 0 \\
    20399.23 & 10078.43 & 3231.40 & 793.25 & 131.33 & 25.95 & 0 & 0 \\
    444615.21 & 221481.03 & 73268.94 & 17791.58 & 3514.66 & 482.12 & 78.49 & 0 \\
    11342675.74 & 5665794.49 & 1883129.50 & 468180.33 & 91361.30 & 15080.79 & 1768.50 & 0 \\
    334032127.30 & 166960507.50 & 55609620.17 & 13867704.58 & 2760946.21 & 449959.01 & 63745.75 & 0 \\
\end{array} \]

- $C_{n,k}$ increase for lower powers of $\ln N$, building up $\sum_{k=1}^{n+1} C_{n,k} \ln^k N \sim n! f_n(N)$

- Truncation at fixed logarithmic accuracy is not a good approximation.

- Renormalon divergence sets in already at low orders — requires a prescription!
**Large–$x$ factorization — going beyond PT**

Perturbation theory (on-shell quark):

$$\Gamma^\text{PT}_N \equiv \int dE_\gamma \left( \frac{2E_\gamma}{m} \right)^{N-1} \frac{1}{\Gamma^\text{PT}_\text{tot}} \frac{d\Gamma^\text{PT}}{dE_\gamma} \approx H(m) J(m^2/N; \mu) S^\text{PT}(m/N; \mu) = H(m) \text{Sud}(N, m)$$

Non-perturbatively (meson):

$$\Gamma_N \equiv \int dE_\gamma \left( \frac{2E_\gamma}{M} \right)^{N-1} \frac{1}{\Gamma^\text{tot}_N} \frac{d\Gamma}{dE_\gamma} \approx H(m) J(m^2/N; \mu) S(M/N; \mu)$$

Quark distribution in the **meson** in terms of that in an **on-shell heavy quark**:

$$S(m/N; \mu) = S^\text{PT}(m/N; \mu) e^{-(N-1)\bar{\Lambda}/M} \mathcal{F}((N-1)\Lambda/M)$$

where $\bar{\Lambda} \equiv M - m$ captures the dependence on the mass; $\mathcal{F}$ has no linear term.
The quark distribution function

Quark distribution \( f(z; \mu) \)
in the **meson** \( (P_B^2 = M^2) \) is defined by

\[
f(z; \mu) \equiv \frac{1}{4\pi} \int_{-\infty}^{\infty} dy^- e^{-izP_B^+y^-} \left\langle B(P_B) \left[ \bar{\Psi}(y) \gamma^+ \Phi_y(0,y) \Psi(0) \right] \right| B(P_B) \rightangle
\]

where \( \Phi_y(0,y) \equiv \textbf{P} \exp \left( ig \int_0^y \! dx^- A^+(x^-) \right) \)

Moments:
\[
F(N; \mu) \equiv \int_0^1 dz z^{N-1} f(z, \mu) \quad \Rightarrow \quad iP_B^+ y^- \longrightarrow N
\]

Perturbative quark distribution \( f_{\text{PT}}(z; \mu) \) in an **on-shell heavy quark** \( (p_b^2 = m^2) \) is defined by the **same operator**, taken between quark states: \( \langle b(p_b) \rangle \).
It is infrared-- and collinear--safe!
From the quark distribution function to power corrections

Quark distribution in an on-shell heavy quark $F_{\mathrm{PT}}(N; \mu)$ and in the meson $F(N; \mu)$:

The two have the same $\mu$ dependence.
They differ by power corrections.

The dependence on the soft scale $m/N$:

\[
F_{\mathrm{PT}}(N; \mu) = \left\langle b(p_b) \left| \left[ \bar{\Psi}(y) \gamma^+ \Phi_y(0, y) \Psi(0) \right]_\mu \right| b(p_b) \right\rangle \bigg|_{i p_b^+ y^- \rightarrow N} + \mathcal{O}(1/N)
\]
\[
= H_F(m) S_{\mathrm{PT}}(m/N; \mu) + \mathcal{O}(1/N)
\]
\[
F(N; \mu) = \left\langle B(P_B) \left| \left[ \bar{\Psi}(y) \gamma^+ \Phi_y(0, y) \Psi(0) \right]_\mu \right| B(P_B) \right\rangle \bigg|_{i P_B^+ y^- \rightarrow N} + \mathcal{O}(1/N)
\]

Using the OPE:

\[
F(N; \mu) = H_F(m) \underbrace{S_{\mathrm{PT}}(m/N; \mu)}_{S(M/N; \mu)} e^{-(N-1)\bar{\Lambda}/M} \mathcal{F}((N-1)\Lambda/M) + \mathcal{O}(1/N),
\]

$\bar{\Lambda} \equiv M - m$ while $\mathcal{F}$ is independent of the quark–mass definition.
$\mathcal{F}$ sums power corrections on the soft scale starting at $\mathcal{O}\left((N-1)\Lambda/M)^2\right)$. 
Renormalon ambiguity in the pole mass

Infrared renormalons: a perturbative probe of large–distance effects

The propagator: \[
\frac{i}{p^2 - m_{\text{MS}}^2 - \Sigma(p, m_{\text{MS}})}
\]

Computed in the large–\(N_f\) limit

Off shell \(\Sigma(p, m_{\text{MS}})\) has no renormalons
But applying the on–shell condition (inverse propagator vanishes at \(p^2 = m^2\)):

\[
\frac{m}{m_{\text{MS}}} = 1 + \frac{C_F}{\beta_0} \int_0^\infty du \left( \frac{\Lambda^2}{m_{\text{MS}}^2} \right)^u \left[ 3e^{5u} \frac{(1 - u)\Gamma(1 + u)\Gamma(-2u)}{\Gamma(3 - u)} + \frac{3}{4u} - R_{\Sigma_1}(u) \right].
\]

Beyond PT the pole mass is ambiguous...

Beneke & Braun; Bigi, Shifman, Uraltsev & Vainshtein (94)

and so is \(\bar{\Lambda} = M - m\).
Cancellation of the leading renormalon ambiguity

The non-perturbative quark distribution function is **renormalon free**:

\[ S(m/N; \mu) \simeq S_{PT}(m/N; \mu) e^{-(N-1)\bar{\Lambda}/M} \]

leading-renormalon cancels out

**Dressed Gluon Exponentiation (DGE):**

\[
S_{PT}(m/N; \mu) = \exp \left\{ \int_{0}^{\infty} du \ T(u) \ \left( \frac{\Lambda^2}{m^2} \right)^u \right. \\
\times \left. \frac{1}{u} \left[ B_S(u) \Gamma(-2u) \left( N^{2u} - 1 \right) + \left( \frac{m^2}{\mu^2} \right)^u B_A(u) \ln N \right] \right\}
\]

\[ B_S(u) = \frac{C_F}{\beta_0} e^{\frac{5}{3}u} (1 - u) + O(1/\beta_0^2); \quad A \text{ is the cusp anomalous dimension.} \]

The renormalon at \( u = \frac{1}{2} \) **cancels** between \( S_{PT}(m/N; \mu) \) and \( e^{-(N-1)\bar{\Lambda}/M} \)!

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Trento, ECT Workshop, May 2005
Renormalons at work — going beyond the large-\(\beta_0\) limit

The cancellation mechanism is general and it can be used:

Calculation of the physical moments using the Principal Value prescription:

\[
\Gamma_N \equiv \int dE_\gamma \left( \frac{2E_\gamma}{M} \right)^{N-1} \frac{d\Gamma}{dE_\gamma}
\]

\[
= H(m) \text{Sud}(N, m)|_{\text{PV}} e^{-\frac{(N-1)\bar{\Lambda}_{\text{PV}}}{M}} \mathcal{F}\left(\frac{(N-1)\Lambda}{M}\right)
\]

where \(\bar{\Lambda}_{\text{PV}} \equiv M - m_{\text{PV}}\);

\(\text{Sud}(N, m)|_{\text{PV}}\) is real–valued: \(\text{Sud}(m, N)|_{\text{PV}} = \left[ \text{Sud}(m, N^*)|_{\text{PV}} \right]^*\).

Calculation of the \(E_\gamma\) spectrum:

\[
\frac{d\Gamma(E_\gamma)}{dE_\gamma} = \frac{M}{2} \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\pi i} \Gamma_N \left( \frac{2E_\gamma}{M} \right)^{-N}
\]

This spectrum is free of the leading renormalon ambiguity!
Sudakov resummation beyond logarithmic accuracy

\[
\text{Sud}(m, N)|_{\text{PV}} = \exp \left\{ \text{PV} \int_0^\infty du T(u) \left( \frac{\Lambda^2}{m^2} \right)^u \right. \\
\times \frac{1}{u} \left[ B_S(u) \Gamma(-2u) (N^{2u} - 1) - B_J(u) \Gamma(-u) (N^u - 1) \right] \}
\]

What do we know about \( B_S(u) \)?

- All orders in the large–\( \beta_0 \) limit: \( B_S(u) = \frac{C_F}{\beta_0} e^{5/2} u (1 - u) + \mathcal{O}(1/\beta_0^2) \)

- NNLO in the full theory: \( B_S(u) = 1 + s_1 \frac{u}{1!} + s_2 \frac{u^2}{2!} + \cdots \)

- Renormalon cancellation in \( \text{Sud}(m, N) e^{-(N-1)\bar{\Lambda}/M} \) implies: \( B_S(u = 1/2) \) is equal in magnitude and opposite in sign to the residue of the \( u = 1/2 \) renormalon in \( m/m_{\overline{\text{MS}}} \), which can be determined from the known NNLO expansion in \( \overline{\text{MS}} \) within a few percent.
Dressed Gluon Exponentiation: Results

\[ \text{Sud}(m, N)|_{PV} = \exp \left\{ \text{PV} \int_0^\infty du \, T(u) \left( \frac{\Lambda^2}{m^2} \right)^u \right\} \times \frac{1}{u} \left[ B_S(u) \Gamma(-2u) \left( N^{2u} - 1 \right) - B_J(u) \Gamma(-u) \left( N^u - 1 \right) \right]. \]

\[ \frac{d\Gamma(E_\gamma)}{dE_\gamma} = \frac{m_{PV}}{2} \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\pi i} \, H(m) \, \text{Sud}(m, N)|_{PV} \left( \frac{2E_\gamma}{m_{PV}} \right)^{-N}. \]

\[ \text{Sud}(N, m)|_{PV} \text{ with various approx. for } B_S(u) \]

Corresponding spectra

Einan Gardi (University of Cambridge) Trento, ECT Workshop, May 2005
Summary

- Resummed perturbation theory predicts the photon energy spectrum in \( \bar{B} \rightarrow X_s \gamma \).

- Renormalon cancellation \((u = 1/2)\) is respected by using the same prescription for both the Sudakov exponent and the pole mass.

- Contrary to Sudakov resummation with fixed logarithmic accuracy, DGE is stable. This is owing to resumming running–coupling effects with a definite prescription.

- The DGE spectrum smoothly extends beyond the perturbative endpoint and tends to zero for \( E_\gamma = (m + \mathcal{O}(\Lambda))/2 \), close to the physical endpoint \( E_\gamma = M/2 \).
Comparison to data: branching fraction

- Theoretical uncertainty on the total BF $\sim 10\%$
- Experimental cuts on $E_\gamma$ do not significantly increase the overall uncertainty.
- The measured BF is consistent with the Standard Model.

- Possible determination of $m_b$!
Comparison to data: cut moments

\[
\langle E_\gamma \rangle_{E_\gamma > E_0} \equiv \frac{1}{\Gamma(E_\gamma > E_0)} \int_{E_0}^{E_\gamma} dE_\gamma \frac{d\Gamma(E_\gamma)}{dE_\gamma} E_\gamma
\]

\[
\left( \langle E_\gamma \rangle_{E_\gamma > E_0} - E_\gamma \right)^n \equiv \frac{1}{\Gamma(E_\gamma > E_0)} \int_{E_0}^{E_\gamma} dE_\gamma \frac{d\Gamma(E_\gamma)}{dE_\gamma} \left( \langle E_\gamma \rangle_{E_\gamma > E_0} - E_\gamma \right)^n.
\]

- Measurement of power corrections.
- Good prospects for determination of $V_{ub}$ from charmless semileptonic decays.