workshop on
"Global Analysis of Polarized RHIC and DIS Data"

Monday, Oct. 8th

Global Analysis Toolbox: Mellin Technique

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the challenge:

analyze an ever growing amount of data
within the best theoretical framework available
in a finite amount of time

a highly non-trivial exercise:

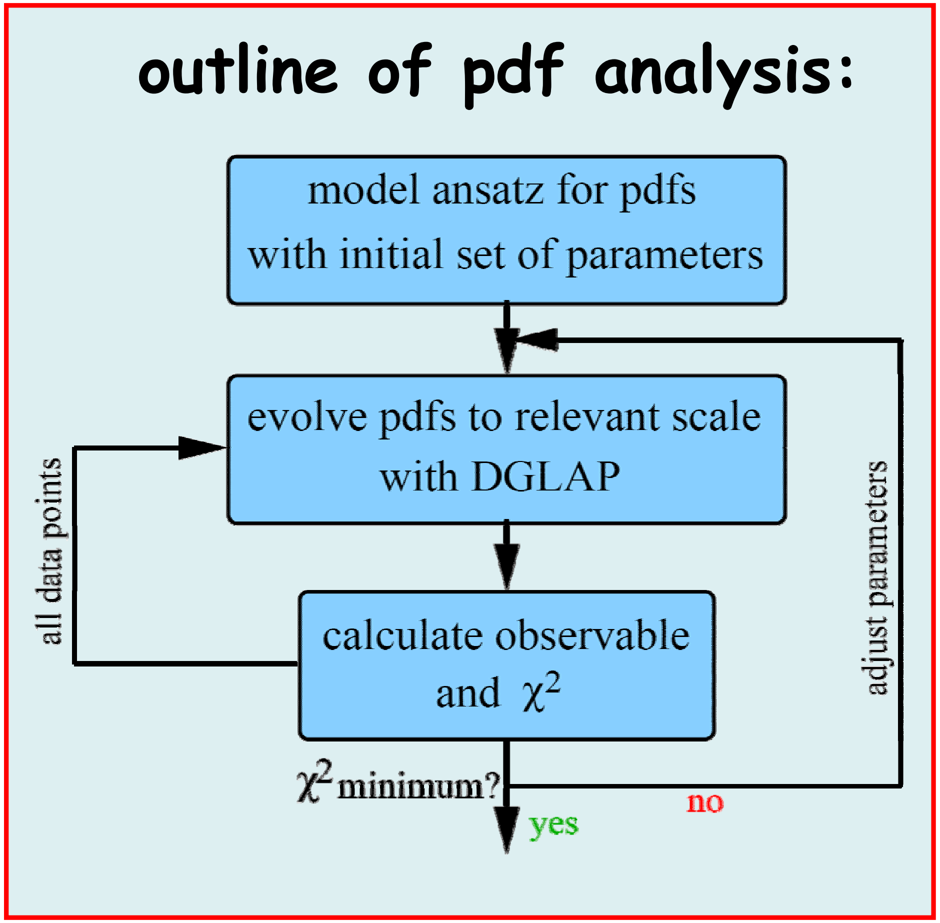
- **recall:** information on spin/nucleon structure and fragmentation cannot be read off from data but is hidden inside complicated convolutions, summed over many subprocesses

$$\boxed{d\Delta\sigma = \sum_{a,b,c} \int \Delta f_a \otimes \Delta f_b \otimes d\Delta\hat{\sigma}_{ab\to cX} \otimes D_c^H}$$

- many different processes needed to pin down all aspects of pdfs/ffct

- all processes must be analyzed in NLO (or beyond) to control theor. errors

the way to do it: global χ^2 minimization



involves multi-parameter fitting

→ **1000's of evaluations of NLO cross sections needed**

(a proper error analysis adds to this)

the problem:

NLO expressions are fairly complex and numerically very time-consuming

computing time for a global analysis at NLO becomes excessive

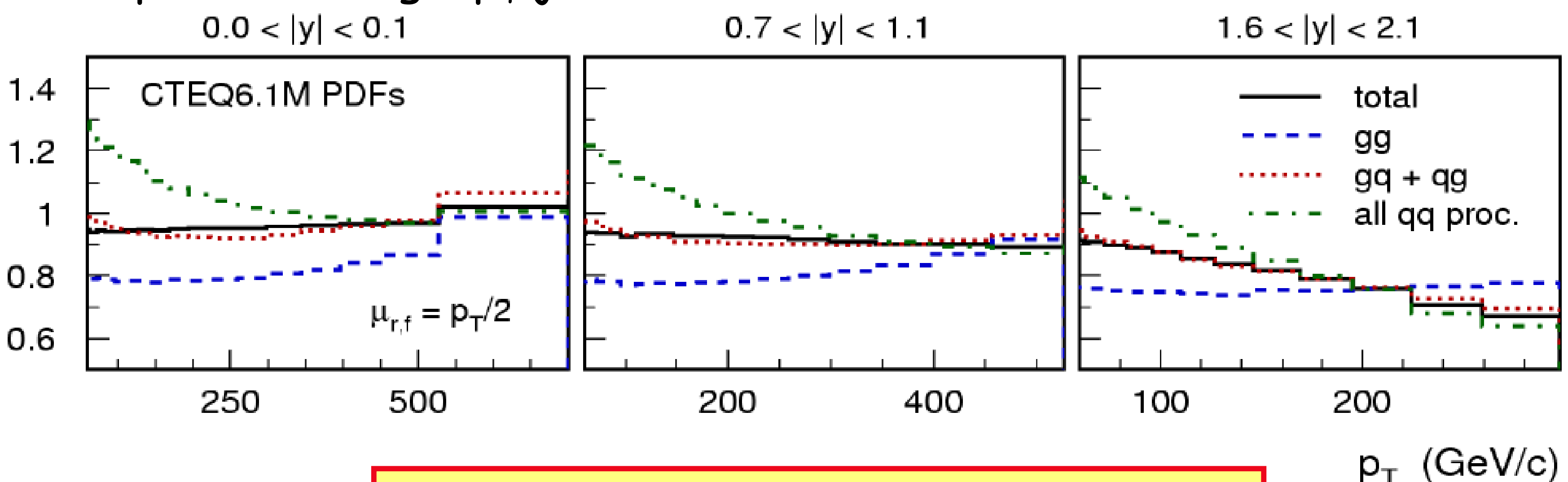
a common "workaround": $d(\Delta)\sigma^{NLO} \approx K \cdot d(\Delta)\sigma^{LO}$

but

- K itself depends on unknown pdfs/ffct
- K = K(p_T,y, ...) and often large
- K different for $d\sigma$ and $d\Delta\sigma$ $\rightarrow$ no cancellation in A_{LL}
- K is different for different subprocesses
- even LO cross sections for pp processes are too slow!

example: K for high-p_T jets at TeVatron

taken from M. Wobisch (fastNLO)



introduces unknown systematic error

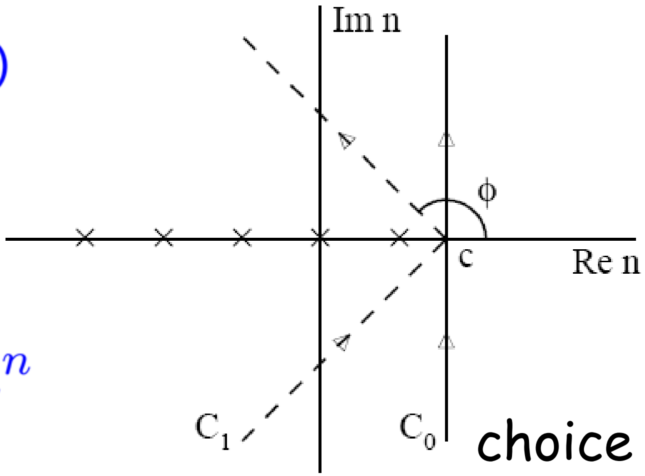
19^{th} century math comes to help …

integral transformation: **Mellin n-moments**

$$h^n \equiv \int_0^1 dx\, x^{n-1}\, h(x)$$

inverse

$$h(x) \equiv \frac{1}{2\pi i} \int_{C_n} dn\, x^{-n}\, h^n$$



choice of contour → later

R.H. Mellin
Finnish mathematician

crucial property: convolutions factorize into simple products

$$\int_0^1 dx\, x^{n-1} \int_x^1 \frac{dy}{y} h(y) g(x/y)$$

$$= \int_0^1 dx\, x^{n-1} \left[\int_0^1 dy \int_0^1 dz\, h(y) g(z) \delta(x - zy) \right]$$

$$= \int_0^1 dy\, y^{n-1} h(y) \int_0^1 dz\, z^{n-1} g(z) = h^n \cdot g^n$$

well-known application: DGLAP evolution

Bjorken-x space: integro-differential equations $\rightarrow$ no analytical solution

e.g. LO non-singlet (valence) evolution

$$\frac{dq(x,Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dy}{y}\, q(y,Q^2) P_{qq}(x/y)$$

known up to NNLO

full singlet evolution more complicated: coupled equations

Mellin-n space: ordinary differential equations $\rightarrow$ analytical solution

$$\frac{dq^n(Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\, q^n(Q^2)\; P_{qq}^n$$

n-moments known analytically

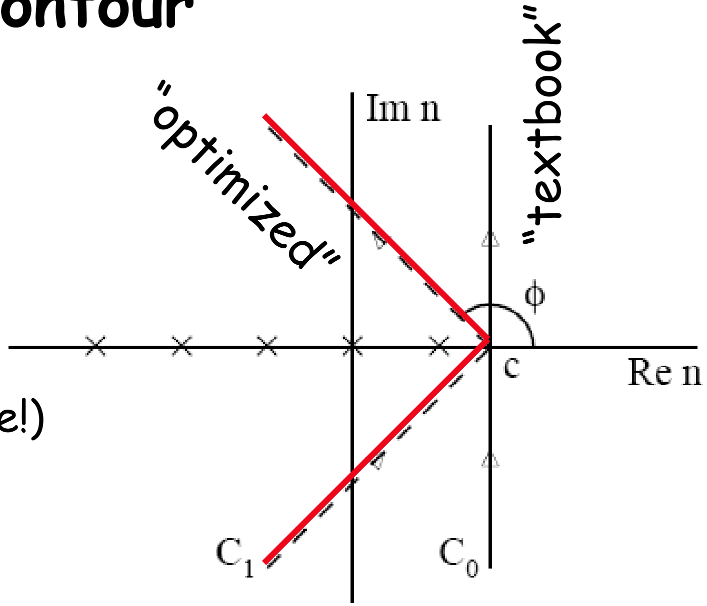
solve analytically (to all orders !):

$$q^n(Q^2) = q^n(Q_0^2)\, \left(\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right)^{-2P_{qq}^n/\beta_0}$$

input (fitted)

pQCD scaling viol.

back into x-space: optimizing the contour



$$f(x,Q^2) \equiv \frac{1}{2\pi i}\int_{\mathcal{C}_n} dn\, x^{-n}\, f^n(Q^2)$$

c: to the right of rightmost pole (convergence!)

f(x) are real functions: $(f^n)^* = f^{n^*}$

$\rightarrow$ parametrize contour to integrate over real variable z

$$f(x,Q^2) = \frac{1}{\pi}\int_0^\infty dz\, \mathrm{Im}\left[e^{-i\phi}\, x^{-c-ze^{i\phi}}\, f^{n=c+ze^{i\phi}}(Q^2)\right]$$

receipe for numerical fast but reliable inversion: see, e.g., A. Vogt "QCD-PEGASUS" code

C_1 contour with $\phi=3\pi/4$: exp. dampening for large |n| $\rightarrow$ can use small z_{max}

in practice: z_{max} adaptive depending on x (large x more costly!)
choose n as supports for Gaussian integration **VERY FAST!**

remarks on different solutions of DGLAP eqs.:

the iterative x-space solution (e.g. Runge-Kutta method) and

the analytical n-space solution may differ in $f(x,Q^2)$!

✓ of course, *differences are only beyond the order considered* but can be non-negligible in comparisons of different codes !!

- iteration introduces more scheme-dep. higher order terms
- n-space solutions truncated at given order satisfy DGLAP eqs. "only" in the sense of a power expansion
- the treatment of the RGE for $\alpha_s(Q^2)$ is also an issue (exact numerical solution vs. power expansions)

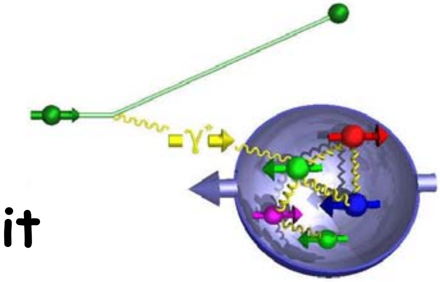
summary so far:

the Q^2 scale evolution of pdfs (and frag. fcts.)
is not an obstacle in a global χ^2 analysis

it can be done to any given order in
a fast and reliable way

but the analyses are on the cross section level …

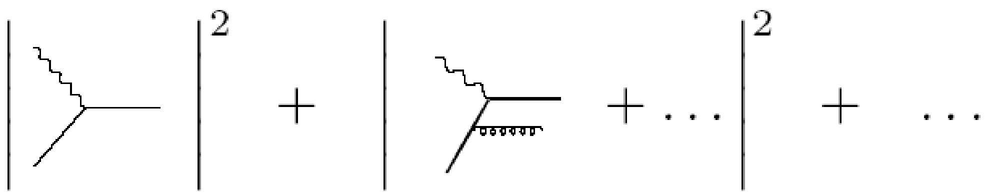
inclusive DIS data ✓



can be included at basically no extra cost in a global fit

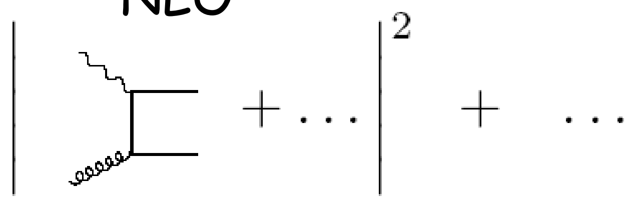
hard scattering coefficient functions known up to NNLO:

$(\Delta)C_q$



LO NLO

$(\Delta)C_g$

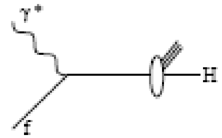


sufficiently "simple" functions: Mellin n-moments can be taken analytically!

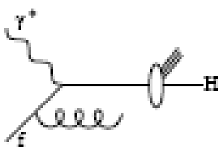
→ $g_1(x,Q^2)$, $F_2(x,Q^2)$, … are obtained by a fast Mellin inversion of their n-moments

$$g_1^n = \frac{1}{2}\sum e_q^2 \Big\{ \Big(1+\frac{\alpha_s}{2\pi}\Delta C_q^n\Big) [\Delta q^n + \Delta \bar{q}^n] + \frac{\alpha_s}{2\pi}\Delta C_g^n \Delta g^n \Big\}$$

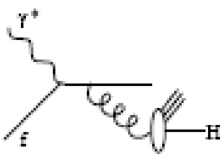
semi-inclusive DIS data



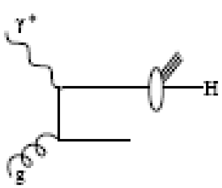
$$\frac{d^3\Delta\sigma}{dx\,dy\,dz} \simeq \sum_{q=u,\bar{u},\ldots,\bar{s}} e_q^2 \left[\overbrace{\Delta q\,(x,\mu_f)\, D_q^H\,(z,\mu_f')} + \int_x^1 \frac{d\hat{x}}{\hat{x}} \int_z^1 \frac{d\hat{z}}{\hat{z}} \left\{ \right.\right.$$



$$\frac{\alpha_s(\mu_r)}{2\pi} \Delta q\left(\frac{x}{\hat{x}},\mu_f\right) \Delta C_{qq}^{(1)}(\hat{x},\hat{z},\mu_f,\mu_f') D_q^H\left(\frac{z}{\hat{z}},\mu_f'\right) +$$



$$\frac{\alpha_s(\mu_r)}{2\pi} \Delta q\left(\frac{x}{\hat{x}},\mu_f\right) \Delta C_{gq}^{(1)}(\hat{x},\hat{z},\mu_f,\mu_f') D_g^H\left(\frac{z}{\hat{z}},\mu_f'\right) +$$



$$\left.\left. \frac{\alpha_s(\mu_r)}{2\pi} \Delta g\left(\frac{x}{\hat{x}},\mu_f\right) \Delta C_{qg}^{(1)}(\hat{x},\hat{z},\mu_f,\mu_f') D_q^H\left(\frac{z}{\hat{z}},\mu_f'\right) \right\} \right]$$

crucial difference: coefficient functions now depend on two variables

"only" NLO corrections are known

nevertheless, Mellin moments can be taken analytically!

but this time we need a **double-Mellin transform**:

Altarelli, Ellis, Martinelli, Pi; MS, Vogelsang

$$\Delta C_{ij}^{nm} \equiv \int_0^1 dx\, x^{n-1} \int_0^1 dz\, z^{m-1}\, \Delta C_{ij}(x,z)$$

and a double Mellin inverse ($\rightarrow$ later!)

in practice, however, experiments integrate over $z_{\text{min}} \gtrsim 0.2$

$\rightarrow$ easier to introduce an "effective" 1-dim. coefficient function

$$\Delta \tilde{C}_j^n \equiv \int_{z_{\text{min}}}^1 dz \int_{\mathcal{C}_m} dm\, z^{-m}\, \Delta C_{ij}^{nm} D_i^m$$

can be pre-calculated *once* prior to the fit and then used like an inclusive DIS coefficient

also **VERY FAST!** ✓

remark on data averaged over x- or z-bins:

can be also directly implemented in moment space:

for instance, $\langle g \rangle(a,b) \equiv \frac{1}{b-a} \int_a^b dx\, g(x)$

insert Mellin representation for g(x) and perform x integration:

$$\langle g \rangle(a,b) = \frac{1}{b-a} \frac{1}{2\pi i} \int_{\mathcal{C}_n} dn\, \frac{b^{1-n} - a^{1-n}}{1-n}\, g^n$$

more efficient than computing g(x) first and then integrating in x-space !

this "trick" was used, e.g., in the global analysis of fragmentation functions (de Florian, Sassot, MS) to include OPAL flavor tagging probabilities it is in the analysis of hadron-hadron data where the double Mellin moment technique exhibits its full potential...

MS, Vogelsang

earlier ideas: Berger, Graudenz, Hampel, Vogt; Kosower

NLO expressions in pp are *much more* complicated than in (SI)DIS

→ Mellin moments cannot be taken analytically & numerically very slow

idea: re-organize multi-convolutions by taking Mellin moments

example: pp→ π X

$$d\Delta\sigma = \sum_{abc} \int \Delta f_a \, \Delta f_b \, d\Delta\hat{\sigma}_{ab\to cX} \, D_c \, dx_a dx_b dz_c$$

express pdfs by their Mellin inverses

$$\frac{1}{2\pi i}\int_{C_n} dn \, x_a^{-n} \, \Delta f_a^n \qquad \frac{1}{2\pi i}\int_{C_m} dm \, x_b^{-m} \, \Delta f_b^m$$

$$= \frac{1}{(2\pi i)^2} \sum_{abc} \underbrace{\int_{C_n} dn \int_{C_m} dm}_{\text{standard Mellin inverse}} \underbrace{\Delta f_a^n \Delta f_b^m}_{\text{fit}} \int x_a^{-n} x_b^{-m} d\Delta\hat{\sigma}_{ab\to cX} \, D_c \, dx_a dx_b dz_c$$

$$\equiv d\Delta\tilde{\sigma}_{ab\to cX}(n,m)$$

can be pre-calculated on grids!

applicability: completely general, tested for pp→γX, pp→πX, pp→jetX

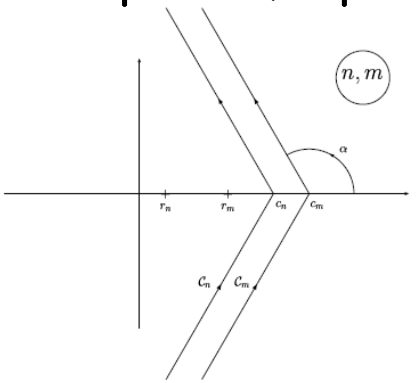
Mellin technique in-depth: the ingredients

$$d\Delta\tilde{\sigma}_{ab\to cX}(n,m)$$

- contains all time-consuming integrations
- calculated once and forever before the fit
- stored in large n×m grids

$$\int_{C_n} dn \int_{C_m} dm$$

- fast numerical Mellin inverse in complex n,m plane
- exponential fall-off of x^{-n}, x^{-m} along contour optimal
- integration = summation in n,m

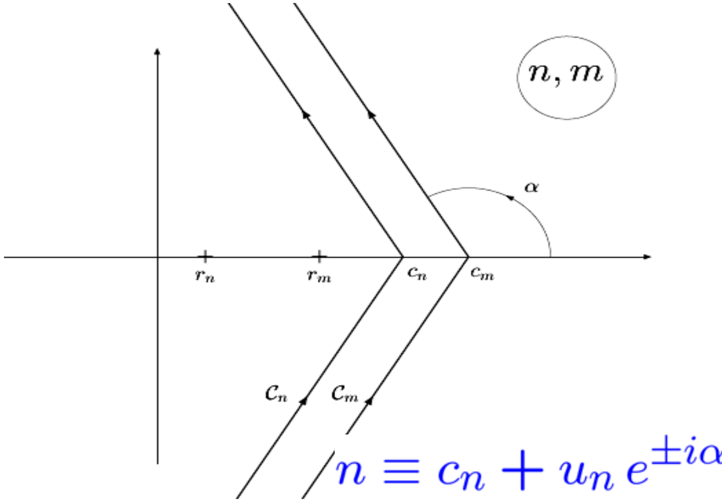


$$\Delta f_a^n \Delta f_b^m$$

- Mellin moments of ansatz for pdfs in x-space, e.g., $f_a(x,\mu_0) = N\, x^{\alpha}\, (1-x)^{\beta}$
- parameters determined in standard χ^2 analysis

Mellin technique in-depth: taking the inverse

straightforward extension to double contour:



use that pdfs are real and parametrize contour with 2 real parameters u_n and u_m

find:

$$d\Delta\sigma = -\frac{1}{2\pi^2} \sum_{a,b} \mathrm{Re} \Big[\int_0^\infty du_n \int_0^\infty du_m\, \Delta f_a^n$$

$$\times \left\{ e^{2i\alpha}\, \Delta f_b^m\, d\Delta\tilde{\sigma}_{ab}(n,m) - (\Delta f_b^m)^*\, d\Delta\tilde{\sigma}_{ab}(n,m^*) \right\} \Big]$$

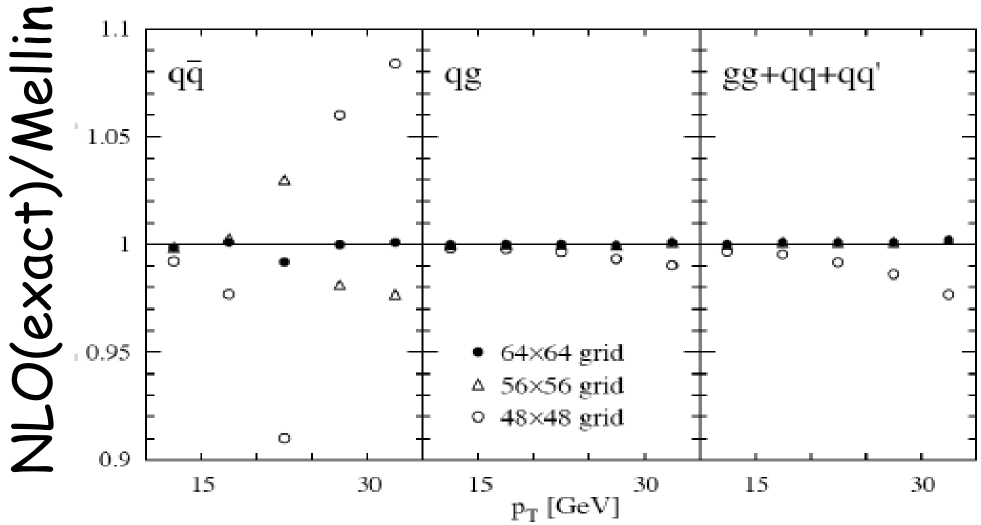
again, choose n,m as supports for Gaussian integration $\rightarrow$ num. fast!

bookkeeping: for *each subprocess* we need two grids: (n,m) and (n,m*)

[in fact this requires four runs (!!) of the NLO codes since -- so far -- they cannot handle complex valued "pdfs" x^{-n}]

Mellin technique in-depth: performance & accuracy

precision: usually, 64×64 grids sufficient for less than 0.5% deviation



example: prompt photons

performance:

"**before**": typ. NLO code O(30sec/p_T value)

"after Mellin tune-up": bullet-train performance

100 evaluations of x-sec take a few seconds

ideal tool for multidim. fitting beyond LO

Mellin technique in-depth: pre-calculated grids

how long does it take?

example: $pp \to \text{jet}\ X$ (to analyse STAR data for A_{LL})

6 different subprocesses to consider: gg, qg, 4* "qq"

$\to 6 \times 4 \times 64 \times 64 \simeq 10^5$ calls of the NLO code per data point

$\to$ current # data points keep 4 dual-core CPU's busy for a week

example: $pp \to \text{hadron}\ X$ (to analyse PHENIX & STAR data for A_{LL})

many more grids since fragmentation distinguishes flavors !!

example: $pp \to \gamma\ X$ (future)

somewhat less demanding than jets ✓

sufficient computing power is essential for the pre-analysis stage

Mellin technique in-depth: improvements

good news: grids only "know about" the kinematics of the exp. bins

→ if binning & kinematics (e.g. η-range) remains unchanged
we can simply add more bins if they become available

Mellin technique has passed an important stress test in
global analyses of fragmentation functions   de Florian, Sassot, MS

improvements: codes not really optimized for speed

parallelization possible

user intervention still required → automatization

currently, changing $\mu_{r,f}$ requires new grids
→ separate grids for the few terms which depend on $\mu_{r,f}$

ready to analyze ...